Representation and Manipulation of Not Necessarily Closed Convex Polyhedra

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PLAN OF THE TALK

Convex Polyhedra: What and Why

What?

- ➜ regions of $\mathbb{R}^n$ bounded by a finite set of hyperplanes.

Why? Solving Classical Data-Flow Analysis Problems!

- ➜ array bound checking and compile-time overflow detection;
- ➜ loop invariant computations and loop induction variables.

Why? Verification of Concurrent and Reactive Systems!

- ➜ synchronous languages;
- ➜ linear hybrid automata (roughly, FSMs with time requirements);
- ➜ systems based on temporal specifications.

And Again: Many Other Applications. . .

- ➜ inferring argument size relationships in logic programs;
- ➜ termination inference for Prolog programs;
- ➜ string cleanness for C programs.

Recent News I

[...] The Mars Climate Orbiter burned in the martian atmosphere in 1999 after missing its orbit insertion because unit computations were inconsistent.

The same year, Mars Polar Lander is suspected of having crashed on Mars upon landing when a software flag was not reset properly.

In [. . .] the 1997 Mars Pathfinder (MPF) technology demonstration mission [. . .] a day's exploration time was lost when ground support teams were forced to reboot the system while downloading science data.

[. . .]

NASA's 2003 Mars Exploration Rover (MER) mission includes two rovers [. . .] At a cost of $400 million for each rover, a coding error that shuts down a rover overnight would in effect be a $4.4 million mistake, as well as a loss of valuable exploration time on the planet.

http://www.arc.nasa.gov/exploringtheuniverse-computercheck.cfm

Recent News II

A previously-unknown software flaw in a widely-deployed General Electric energy management system contributed to the devastating scope of the August 14th northeastern U.S. blackout, industry officials revealed this week.

The bug in GE Energy's XA/21 system was discovered in an intensive code audit conducted by GE and a contractor in the weeks following the blackout, according to FirstEnergy Corp., the Ohio utility where investigators say the blackout began. "It had never evidenced itself until that day," said spokesman Ralph DiNicola. "This fault was so deeply embedded, it took them weeks of poring through millions of lines of code and data to find it."

[. . .]

The cascading blackout eventually cut off electricity to 50 million people in eight states and Canada.

http://www.securityfocus.com/news/8016

The Double Description Method by Motzkin et al.

Constraint Representation

- If $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \neq \mathbf{0}$, and $b \in \mathbb{R}$, the linear inequality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$ defines a closed affine half-space.
- All *closed polyhedra* can be expressed as the conjunction of a finite number of such constraints.

Generator Representation

- If $\mathcal{P} \subseteq \mathbb{R}^n$, a *point of* $\mathcal{P}$ is any $\boldsymbol{p} \in \mathcal{P}$.
- If $\mathcal{P} \subseteq \mathbb{R}^n$ and $\mathcal{P} \neq \varnothing$, a vector $\boldsymbol{r} \in \mathbb{R}^n$ such that $\boldsymbol{r} \neq \mathbf{0}$ is a *ray of* $\mathcal{P}$ iff for each point $\boldsymbol{p} \in \mathcal{P}$ and each $\lambda \in \mathbb{R}_+$, we have $\boldsymbol{p} + \lambda \boldsymbol{r} \in \mathcal{P}$.
- All *closed polyhedra* can be expressed as

$$\left\{ R\boldsymbol{\rho} + P\boldsymbol{\pi} \in \mathbb{R}^n \;\middle|\; \boldsymbol{\rho} \in \mathbb{R}_+^r, \boldsymbol{\pi} \in \mathbb{R}_+^p, \textstyle\sum_{i=1}^p \pi_i = 1 \right\}$$

where $R \in \mathbb{R}^{n \times r}$ is a matrix having rays of the polyhedron as columns and $P \in \mathbb{R}^{n \times p}$ has points of the polyhedron for its columns.

The Double Description Method (cont'd)

Constraint Representation

➜ Special case: $n = 0$ and $\mathcal{P} = \varnothing$.

➜ The equality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = b$ defines an affine hyperplane...

  ➜ ...that is equivalent to the pair $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$ and $\langle -\boldsymbol{a}, \boldsymbol{x} \rangle \geq -b$.

➜ If $\mathcal{C}$ is a finite set of constraints we call it *a system of constraints* and write $\mathrm{con}(\mathcal{C})$ to denote the polyhedron it describes.

Generator Representation

➜ Note: $P = \varnothing$ if and only if $\mathcal{P} = \varnothing$.

➜ Note: points are not necessarily vertices and rays are not necessarily extreme.

➜ We call $\mathcal{G} = (R, P)$ *a system of generators* and write $\mathrm{gen}(\mathcal{G})$ to denote the polyhedron it describes.

DD Pairs and Minimality

Representing a Polyhedron Both Ways

- Let $\mathcal{P} \subseteq \mathbb{R}^n$. If $\mathrm{con}(\mathcal{C}) = \mathrm{gen}(\mathcal{G}) = \mathcal{P}$, then $(\mathcal{C}, \mathcal{G})$ is said to be a DD pair for $\mathcal{P}$.

Minimality of the Representations

- $\mathcal{C}$ is in minimal form if there does not exist $\mathcal{C}' \subset \mathcal{C}$ such that $\mathrm{con}(\mathcal{C}') = \mathcal{P}$;
- $\mathcal{G} = (R, P)$ is in minimal form if there does not exist $\mathcal{G}' = (R', P') \neq \mathcal{G}$ such that $R' \subseteq R$, $P' \subseteq P$ and $\mathrm{gen}(\mathcal{G}') = \mathcal{P}$;
- the DD pair $(\mathcal{C}, \mathcal{G})$ is in minimal form if $\mathcal{C}$ and $\mathcal{G}$ are both in minimal form.

But, wait a minute…

…why keeping two representations for the same object?

Advantages of the Dual Description Method

Some Operations Are More Efficiently Performed on Constraints

- ➜ Intersection is implemented as the union of constraint systems.
- ➜ Adding constraints (of course).
- ➜ Relation polyhedron-generator (subsumes or not).

Some Operations Are More Efficiently Performed on Generators

- ➜ Convex polyhedral hull (poly-hull): union of generator systems.
- ➜ Adding generators (of course).
- ➜ Projection (i.e., removing dimensions).
- ➜ Relation polyhedron-constraint (disjoint, intersects, includes . . .).
- ➜ Finiteness (boundedness) check.
- ➜ Time-elapse.

Some Operations Are More Efficiently Performed with Both

- ➜ Inclusion and equality tests.
- ➜ Widening.

FURTHER ADVANTAGES OF THE DUAL DESCRIPTION METHOD

The Principle of Duality

- ➜ Systems of constraints and generators enjoy a quite strong and useful duality property.
- ➜ Very roughly speaking:
  - ➜ the constraints of a polyhedron are (almost) the generators of the *polar* of the polyhedron;
  - ➜ the generators of a polyhedron are (almost) the constraints of the polar of the polyhedron;
  - ➜ the polar of the polar of a polyhedron is the polyhedron itself.

$\Longrightarrow$ Computing constraints from generators is the same problem as computing generators from constraints.

The Algorithm of Motzkin-Chernikova-Le Verge

- ➜ Solves both problems yielding a minimized system...
- ➜ ...and can be implemented so that the source system is also minimized in the process.

Strict Inequalities and NNC Polyhedra

- If $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \neq \mathbf{0}$, and $b \in \mathbb{R}$, the linear strict inequality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b$ defines an open affine half-space;
- when strict inequalities are allowed in the system of constraints we have polyhedra that are not necessarily closed: NNC polyhedra.

Encoding NNC Polyhedra as C Polyhedra

- call $\mathbb{P}_n$ and $\mathbb{CP}_n$ the sets of all NNC and closed polyhedra, respectively;
- each NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ can be embedded into a closed polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$:
- the additional dimension of the vector space, usually labeled by the letter ϵ, encodes the topological closedness of each affine half-space in the constraint description for $\mathcal{P}$.

Embedding $\mathbb{P}_n$ Into $\mathbb{CP}_{n+1}$: Halbwachs et al.

If $\mathcal{P} \in \mathbb{P}_n$ and $\mathcal{P} = \mathrm{con}(\mathcal{C})$, where

$$\mathcal{C} = \bigl\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \bowtie_i b_i \bigm| i \in \{1, \ldots, m\}, \boldsymbol{a}_i \in \mathbb{R}^n, \bowtie_i \in \{\geq, >\}, b_i \in \mathbb{R} \bigr\},$$

then $\mathcal{R} \in \mathbb{CP}_{n+1}$ is defined by $\mathcal{R} = \mathrm{con}\bigl(\mathrm{con\_repr}(\mathcal{C})\bigr)$, where

$$\begin{aligned} \mathrm{con\_repr}(\mathcal{C}) \stackrel{\mathrm{def}}{=} & \bigl\{0 \leq \epsilon \leq 1\bigr\} \\ & \cup \bigl\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle - 1 \cdot \epsilon \geq b_i \bigm| i \in \{1, \ldots, m\}, \bowtie_i \in \{>\} \bigr\} \\ & \cup \bigl\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b_i \bigm| i \in \{1, \ldots, m\}, \bowtie_i \in \{\geq\} \bigr\}. \end{aligned}$$

What Are the Generators of NNC Polyhedra

- A fundamental feature of the DD method: the ability to represent polyhedra both by constraints and generators.
- But what are the generators for NNC polyhedra?
- From the New Polka manual (s is the ϵ coefficient):

  Don't ask me the intuitive meaning of $s \neq 0$ in rays and vertices !

- From the Polka manual:

  While strict inequations handling is transparent for constraints [...] the extra dimension added to the variables space is apparent when it comes to generators [...]
  This makes more difficult to define polyhedra with the only help of generators : one should carefully study the extra vertices with non null `epsilon` coefficients added to constraints defined polyhedra [...]

Closure Points to the Rescue

- By decoupling the user interface from the details of the particular implementation, it is possible to provide an intuitive generalization of the concept of generator system.
- The key step is the introduction of a new kind of generators: closure points:
  - a vector $\boldsymbol{c} \in \mathbb{R}^n$ is a *closure point* of $S \subseteq \mathbb{R}^n$ if and only if $\boldsymbol{c} \in \mathbb{C}(S)$.
- Characterization of closure points for NNC polyhedra:
  - a vector $\boldsymbol{c} \in \mathbb{R}^n$ is a closure point of the NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ if and only if $\mathcal{P} \neq \varnothing$ and for every point $\boldsymbol{p} \in \mathcal{P}$ and $\lambda \in \mathbb{R}$ such that $0 < \lambda < 1$, it holds $\lambda \boldsymbol{p} + (1 - \lambda)\boldsymbol{c} \in \mathcal{P}$.
- All *NNC polyhedra* can be expressed as

$$\left\{ R\boldsymbol{\rho} + P\boldsymbol{\pi} + C\boldsymbol{\gamma} \in \mathbb{R}^n \;\middle|\; \boldsymbol{\rho} \in \mathbb{R}^r_+, \boldsymbol{\pi} \in \mathbb{R}^p_+, \boldsymbol{\pi} \neq \boldsymbol{0}, \boldsymbol{\gamma} \in \mathbb{R}^c_+, \sum_{i=1}^p \pi_i + \sum_{i=1}^c \gamma_i = 1 \right\}$$

where $R \in \mathbb{R}^{n \times r}$ is a matrix having rays of the polyhedron as columns, $P \in \mathbb{R}^{n \times p}$ has points of the polyhedron for its columns, and $C \in \mathbb{R}^{n \times c}$ has closure points of the polyhedron for its columns.

Not Necessarily Closed Polyhedra: Take Two

Constraint Representation: $\mathrm{con}(\mathcal{C})$

- ➜ If $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \neq \mathbf{0}$, and $b \in \mathbb{R}$, the linear non-strict (resp., strict) inequality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$ (resp., $\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b$) defines a closed (resp., open) affine half-space.
- ➜ Mixed constraint systems $\Longleftrightarrow$ NNC polyhedra.

Not Necessarily Closed Polyhedra: Take Two

Constraint Representation: $\mathrm{con}(\mathcal{C})$

- If $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \neq \boldsymbol{0}$, and $b \in \mathbb{R}$, the linear non-strict (resp., strict) inequality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$ (resp., $\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b$) defines a closed (resp., open) affine half-space.
- Mixed constraint systems $\Longleftrightarrow$ NNC polyhedra.

Generator Representation: $\mathrm{gen}(\mathcal{G})$, where $\mathcal{G} = (R, P, C)$

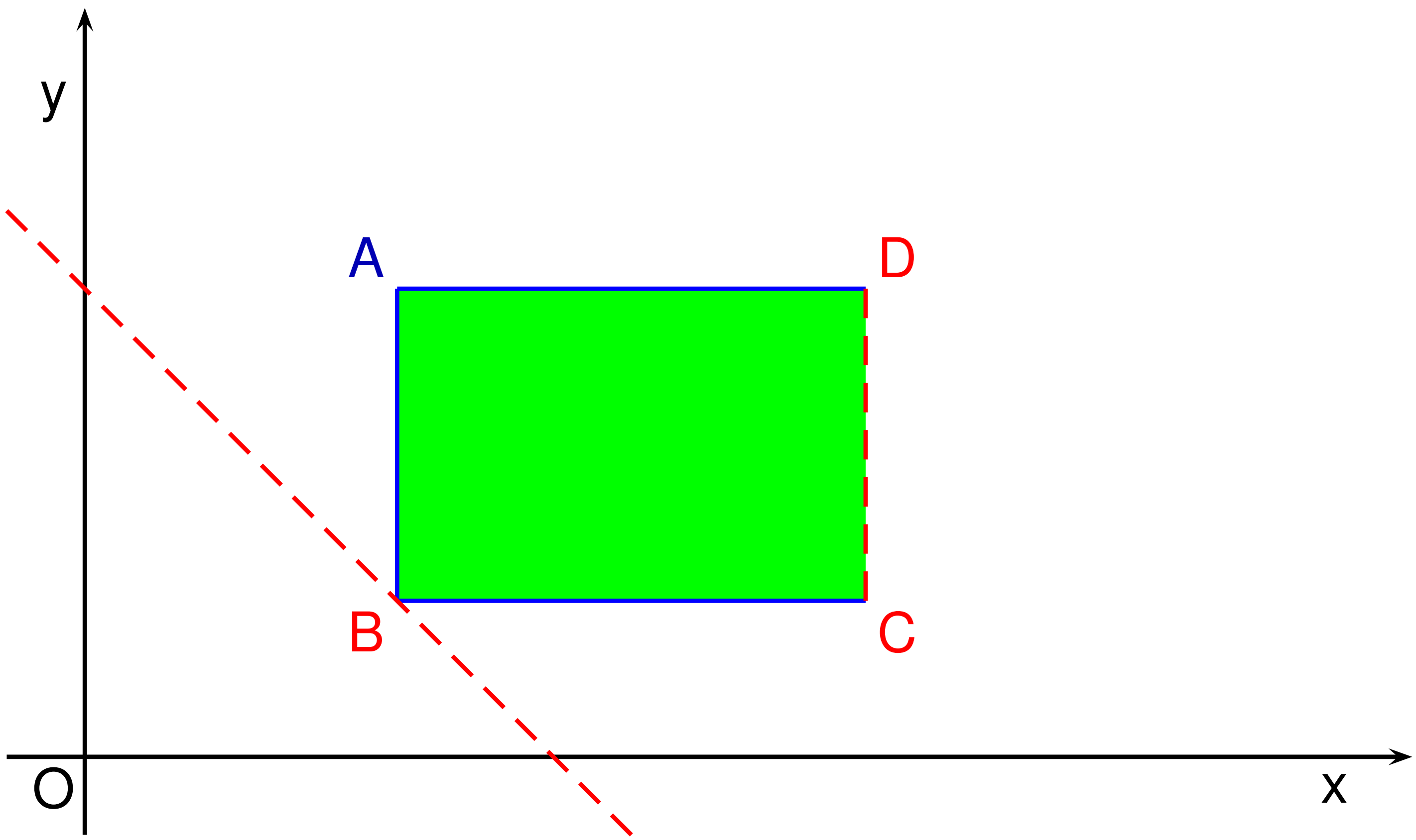
- $\boldsymbol{r} \in \mathbb{R}^n$ is a ray of $\mathcal{P} \subseteq \mathbb{R}^n$ iff it is a direction of infinity for $\mathcal{P}$;
- $\boldsymbol{p} \in \mathbb{R}^n$ is a point of $\mathcal{P} \subseteq \mathbb{R}^n$ iff $\boldsymbol{p} \in \mathcal{P}$.
- $\boldsymbol{c} \in \mathbb{R}^n$ is a closure point of $\mathcal{P} \subseteq \mathbb{R}^n$ iff $\boldsymbol{c} \in \mathbb{C}(\mathcal{P})$.
- All NNC polyhedra can be expressed as

$$\left\{ R\boldsymbol{\rho} + P\boldsymbol{\pi} + C\boldsymbol{\gamma} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \boldsymbol{\rho} \in \mathbb{R}^r_+, \boldsymbol{\pi} \in \mathbb{R}^p_+, \boldsymbol{\gamma} \in \mathbb{R}^c_+, \\ \boldsymbol{\pi} \neq \boldsymbol{0}, \sum_{i=1}^p \pi_i + \sum_{i=1}^c \gamma_i = 1 \end{array} \right\}.$$

- Extended generator systems $\Longleftrightarrow$ NNC polyhedra.

EXAMPLE USING CONSTRAINTS

$\mathcal{P} = \mathrm{con}\big(\{2 \leq x, x < 5, 1 \leq y \leq 3, x + y > 3\}\big).$

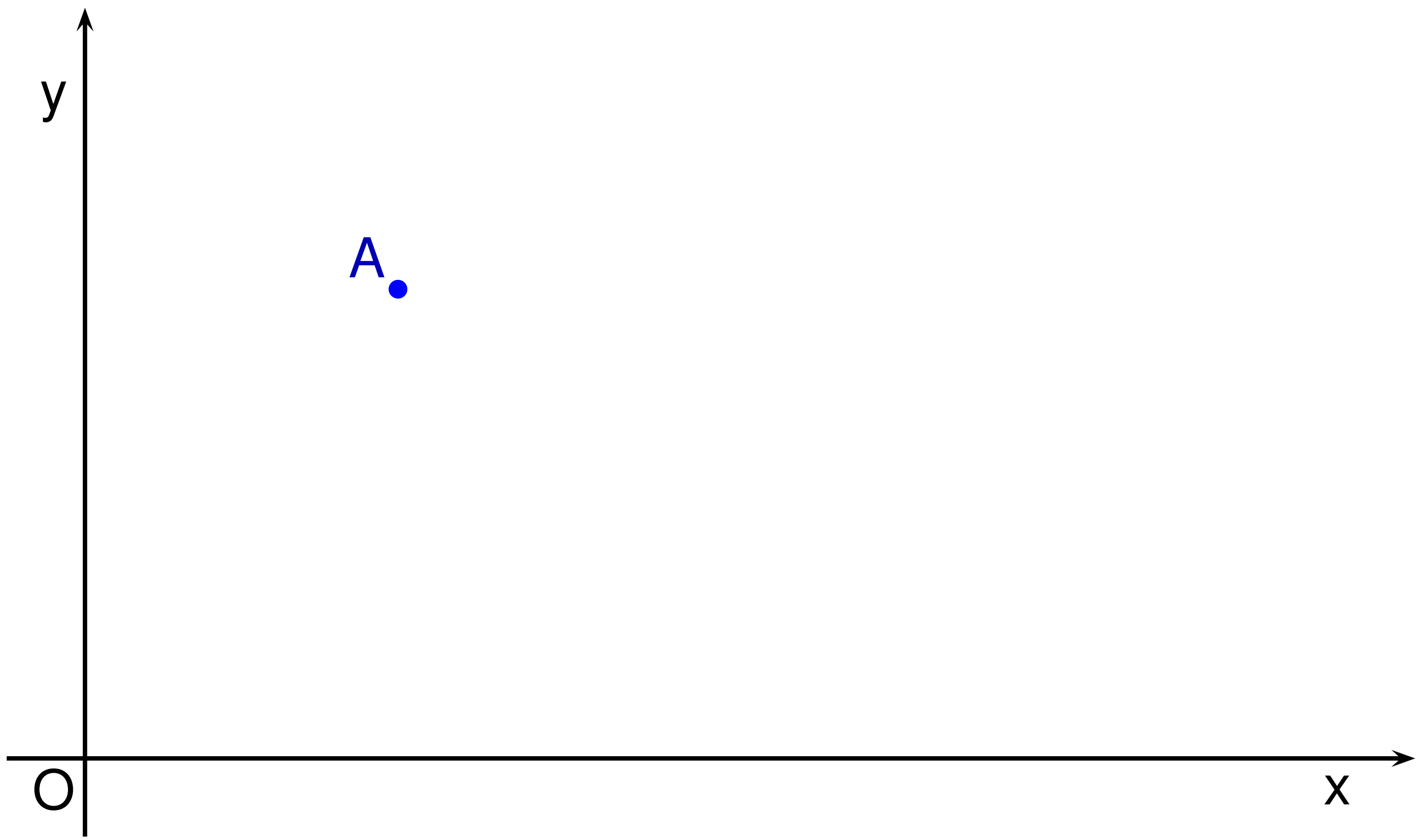


Same Example Using Generators (i)

$\mathcal{P} = \mathrm{gen}\big((R, P, C)\big) = \mathrm{gen}\big((\varnothing, \varnothing, \varnothing)\big)$.

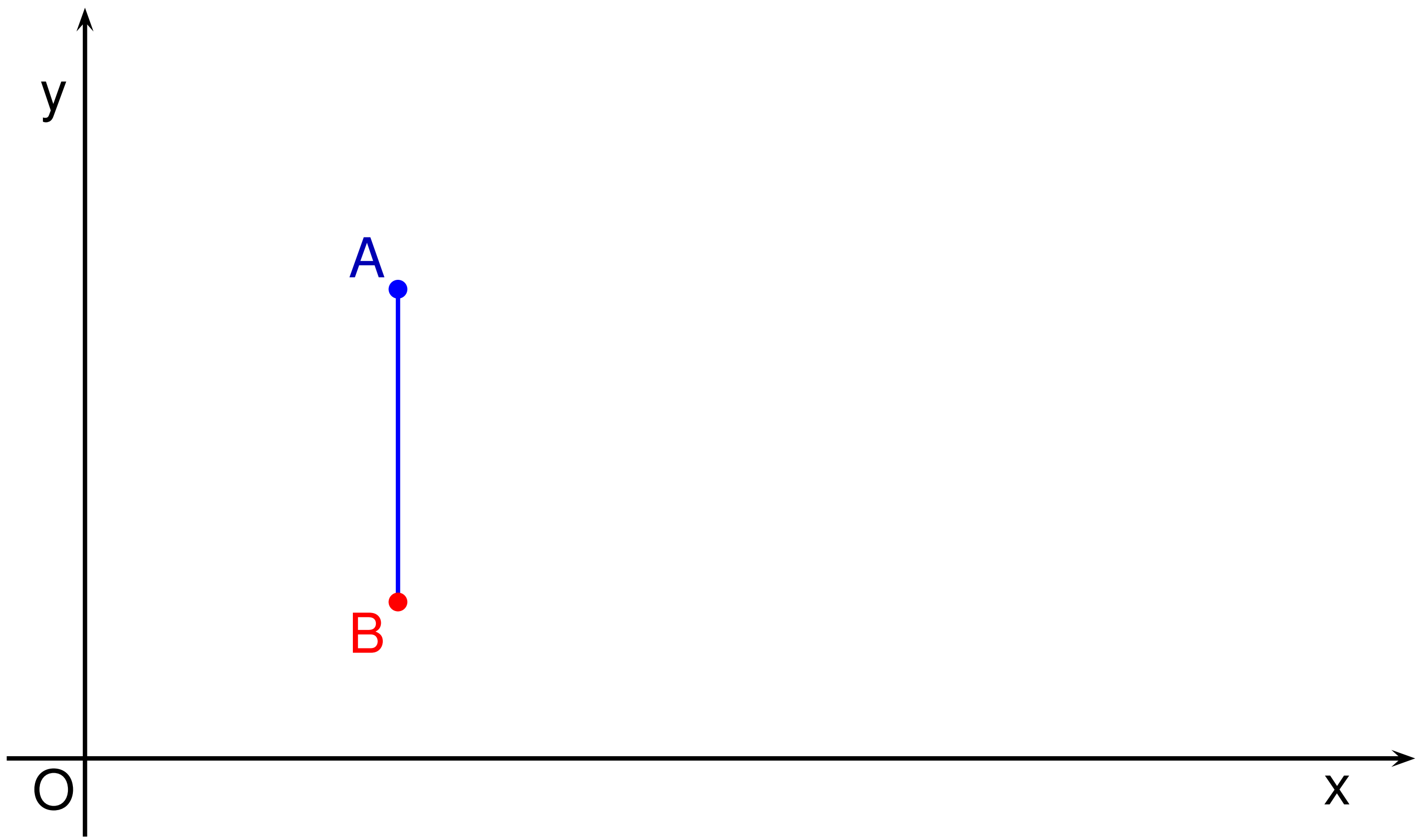
Same Example Using Generators (II)

$$\mathcal{P} = \mathrm{gen}\big((R, P, C)\big) = \mathrm{gen}\Big(\big(\varnothing, \{A\}, \varnothing\big)\Big).$$



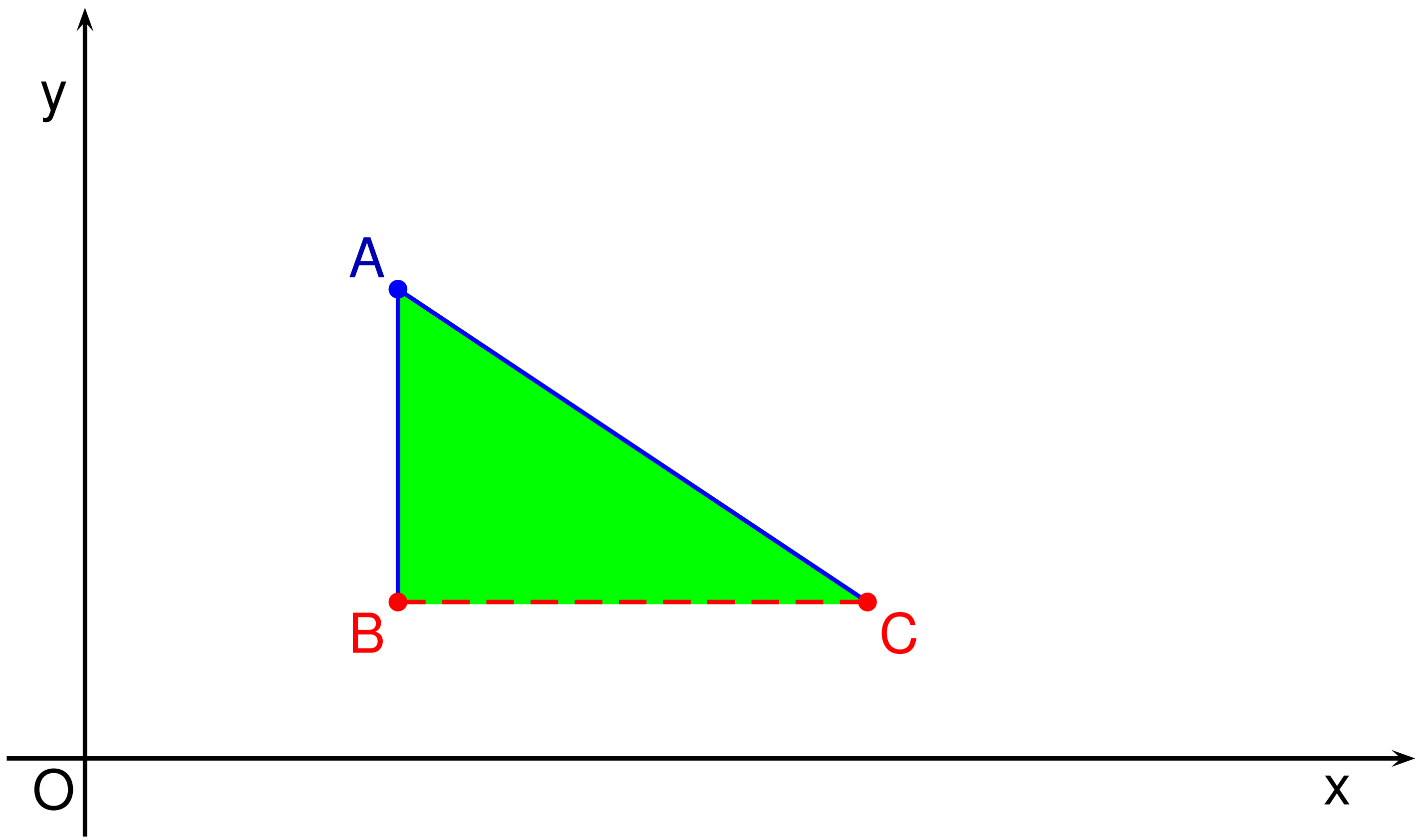
Same Example Using Generators (iii)

$$\mathcal{P} = \text{gen}\big((R, P, C)\big) = \text{gen}\Big(\big(\varnothing, \{A\}, \{B\}\big)\Big).$$



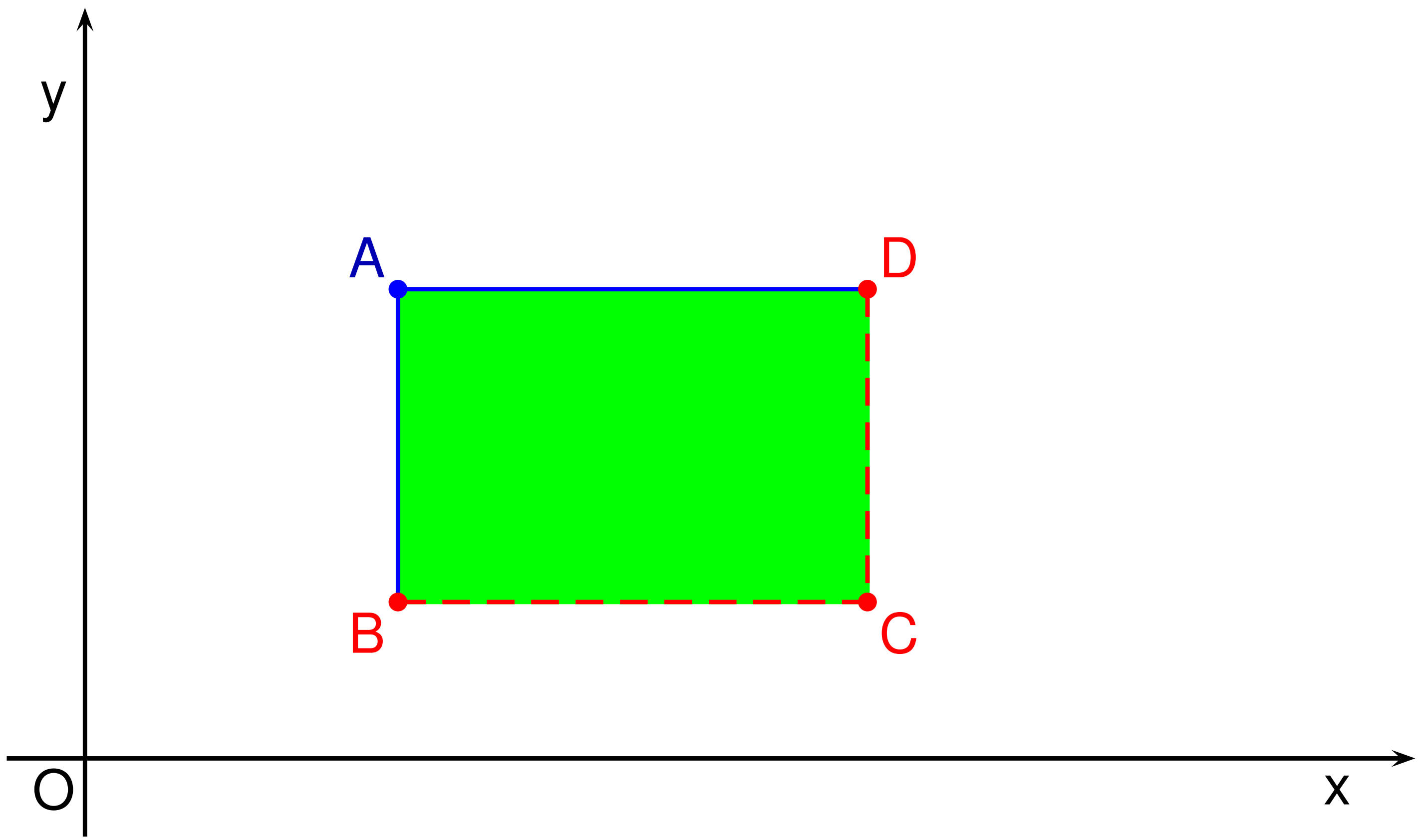
Same Example Using Generators (iv)

$$\mathcal{P} = \text{gen}\big((R, P, C)\big) = \text{gen}\Big(\big(\emptyset, \{A\}, \{B, C\}\big)\Big).$$



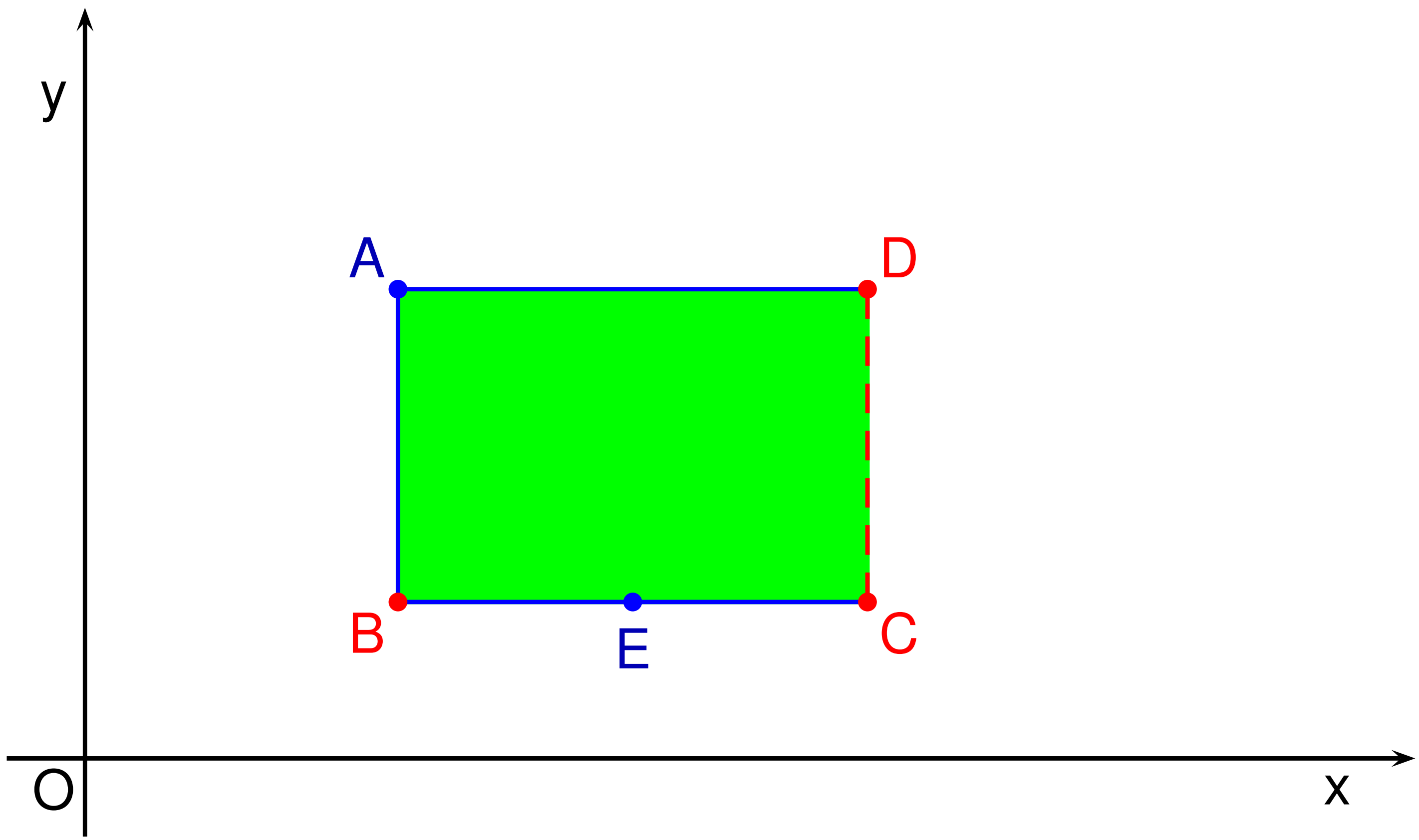
Same Example Using Generators (v)

$$\mathcal{P} = \mathrm{gen}\big((R, P, C)\big) = \mathrm{gen}\Big(\big(\varnothing, \{A\}, \{B, C, D\}\big)\Big).$$



Same Example Using Generators (vi)

$$\mathcal{P} = \mathrm{gen}\big((R, P, C)\big) = \mathrm{gen}\Big(\big(\varnothing, \{A, E\}, \{B, C, D\}\big)\Big).$$



Encoding NNC Polyhedra as C Polyhedra

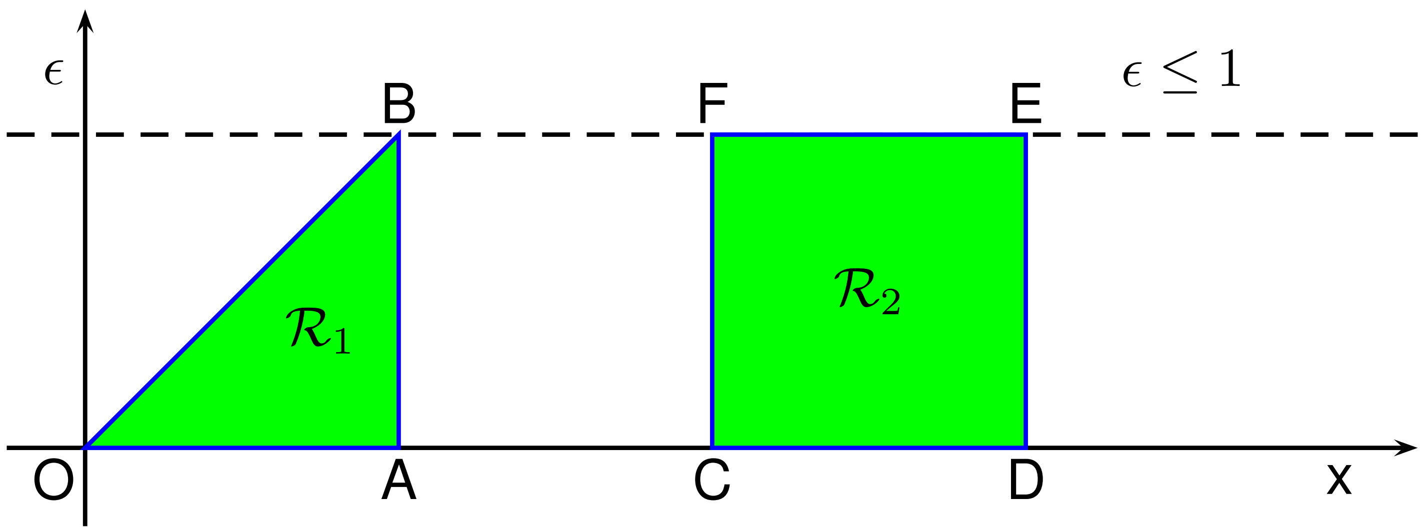
- Let $\mathbb{P}_n$ and $\mathbb{CP}_n$ be the sets of all NNC and closed polyhedra, respectively: each $\mathcal{P} \in \mathbb{P}_n$ can be embedded into $\mathcal{R} \in \mathbb{CP}_{n+1}$.
- A new dimension is added, the ϵ coordinate:
  - to distinguish between strict and non-strict constraints;
  - to distinguish between points and closure points.
- (Will denote by e the coefficient of the ϵ coordinate.)
- The encoded NNC polyhedron:

$$\mathcal{P} = [\![\mathcal{R}]\!] \stackrel{\text{def}}{=} \left\{\, \boldsymbol{v} \in \mathbb{R}^n \;\middle|\; \exists e > 0 \,.\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R} \,\right\}.$$

EXAMPLE: ENCODING $\mathbb{P}_1$ INTO $\mathbb{CP}_2$

$\mathcal{R}_1$ encodes $\mathcal{P}_1 = \mathrm{con}\big(\{0 < x \leq 1\}\big)$,

$\mathcal{R}_2$ encodes $\mathcal{P}_2 = \mathrm{con}\big(\{2 \leq x \leq 3\}\big)$.



THE APPROACH BY HALBWACHS ET AL. REVISITED

- If $\mathcal{P} \in \mathbb{P}_n$ and $\mathcal{P} = \mathrm{con}(\mathcal{C})$, where

$$\mathcal{C} = \left\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \bowtie_i b_i \;\middle|\; i \in \{1, \ldots, m\}, \boldsymbol{a}_i \in \mathbb{R}^n, \bowtie_i \in \{\geq, >\}, b_i \in \mathbb{R} \right\},$$

then $\mathcal{R} \in \mathbb{CP}_{n+1}$ is defined by $\mathcal{R} = \mathrm{con}\bigl(\mathrm{con\_repr}(\mathcal{C})\bigr)$, where

$$\begin{aligned} \mathrm{con\_repr}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \bigl\{0 \leq \epsilon \leq 1\bigr\} & \\ & \cup \left\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle - 1 \cdot \epsilon \geq b_i \;\middle|\; i \in \{1, \ldots, m\}, \bowtie_i \in \{>\} \right\} \\ & \cup \left\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b_i \;\middle|\; i \in \{1, \ldots, m\}, \bowtie_i \in \{\geq\} \right\}. \end{aligned}$$

- If $\mathcal{P} \in \mathbb{P}_n$ and $\mathcal{P} = \mathrm{gen}(\mathcal{G})$, where $\mathcal{G} = (R, P, C)$, then $\mathcal{R} \in \mathbb{CP}_{n+1}$ is defined by $\mathcal{R} = \mathrm{gen}\bigl(\mathrm{gen\_repr}(\mathcal{G})\bigr) = \mathrm{gen}\bigl((R', P')\bigr)$, where

$$R' = \left\{ (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\middle|\; \boldsymbol{r} \in R \right\},$$

$$P' = \left\{ (\boldsymbol{p}^{\mathrm{T}}, 1)^{\mathrm{T}}, (\boldsymbol{p}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\middle|\; \boldsymbol{p} \in P \right\} \cup \left\{ (\boldsymbol{c}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\middle|\; \boldsymbol{c} \in C \right\}.$$

THE APPROACH BY HALBWACHS ET AL. (CONT'D)

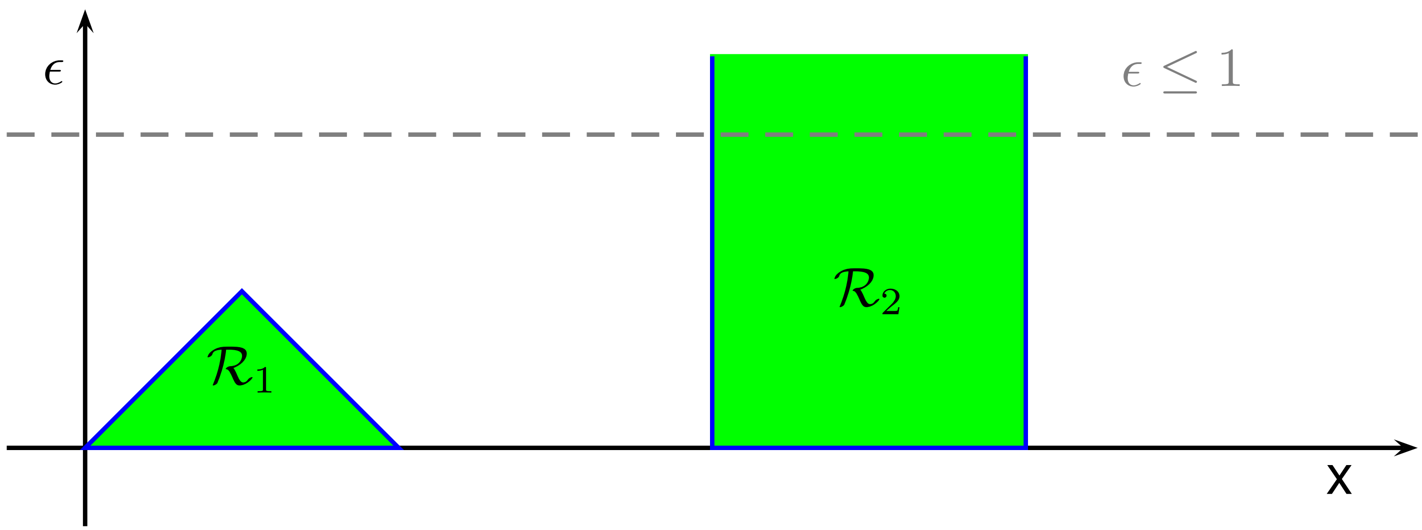
- With a little precaution the operations on representations do (or can be slightly modified to do) what is expected:
  - intersection;
  - convex polyhedral hull;
  - affine image and preimage;
  - ...
- This encoding is used in the New Polka library by B. Jeannet and in the Parma Polyhedra Library.
- Is this approach the only possible one?
- Can we generalize this construction so as to preserve its good qualities?

THE CONSTRAINT $\epsilon \leq \delta$ IS NEEDED ...

Suppose we do not add any ϵ-upper-bound constraint:

$\mathcal{R}_1$ encodes $\mathcal{P}_1 = \mathrm{con}\big(\{0 < x < 1\}\big)$,

$\mathcal{R}_2$ encodes $\mathcal{P}_2 = \mathrm{con}\big(\{2 \leq x \leq 3\}\big)$.

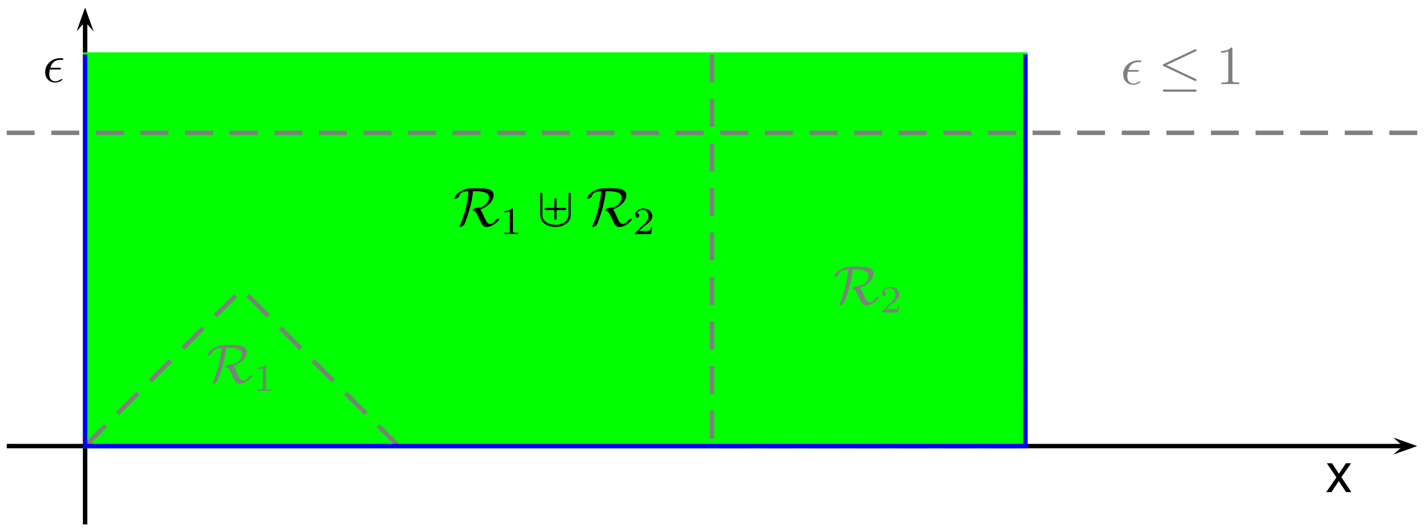


. . . Because Otherwise the Poly-Hull is Not Correct

The poly-hull $\mathcal{P}_1 \uplus \mathcal{P}_2$ is not represented correctly by $\mathcal{R}_1 \uplus \mathcal{R}_2$.

$$\mathcal{P}_1 \uplus \mathcal{P}_2 \stackrel{\text{def}}{=} \operatorname{con}\big(\{0 < x \leq 3\}\big),$$

$\mathcal{R}_1 \uplus \mathcal{R}_2$ encodes $\mathcal{P}' = \operatorname{con}\big(\{0 \leq x \leq 3\}\big)$.

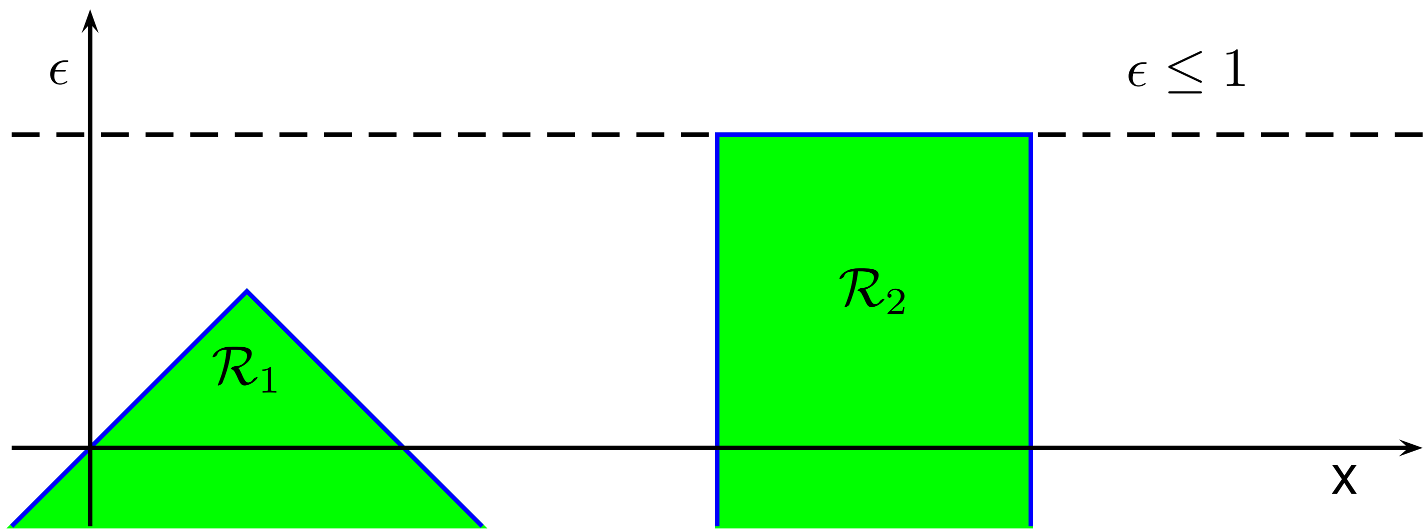


THE CONSTRAINT $\epsilon \geq 0$ IS NEEDED . . .

Suppose we do not add the non-negativity constraint for ϵ:

$\mathcal{R}_1$ encodes $\mathcal{P}_1 = \mathrm{con}\big(\{0 < x < 1\}\big)$,

$\mathcal{R}_2$ encodes $\mathcal{P}_2 = \mathrm{con}\big(\{2 \leq x \leq 3\}\big)$.

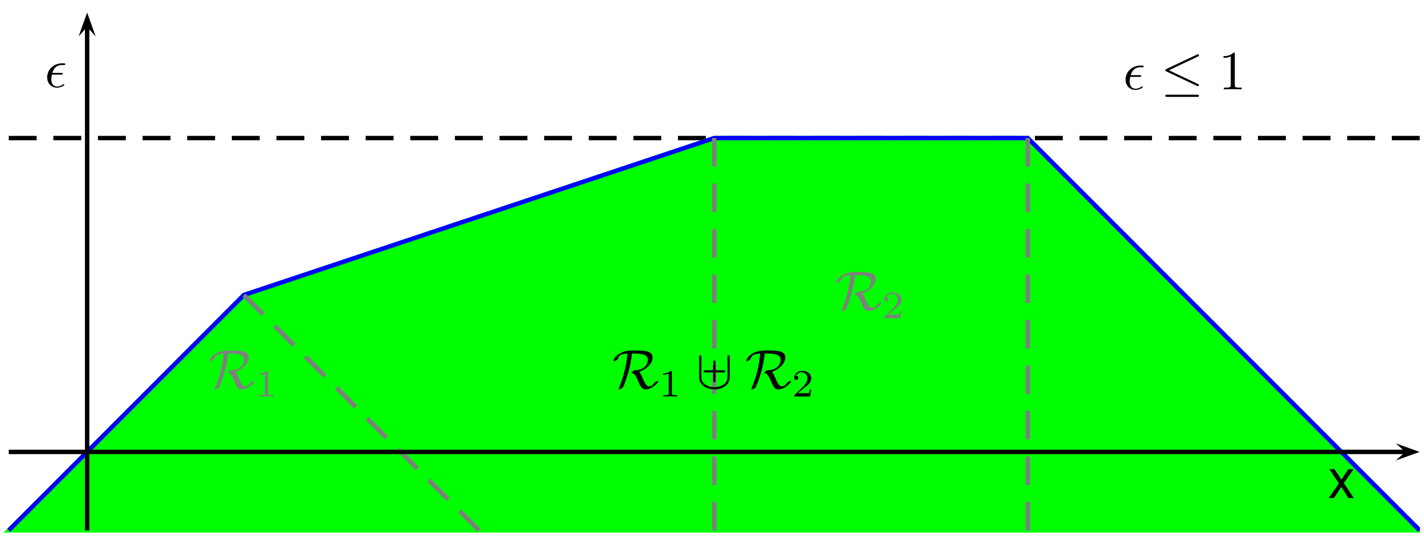


. . . For the Same Reason . . .

The poly-hull $\mathcal{P}_1 \uplus \mathcal{P}_2$ is not represented correctly by $\mathcal{R}_1 \uplus \mathcal{R}_2$.

$$\mathcal{P}_1 \uplus \mathcal{P}_2 \stackrel{\text{def}}{=} \operatorname{con}\bigl(\{0 < x \leq 3\}\bigr),$$

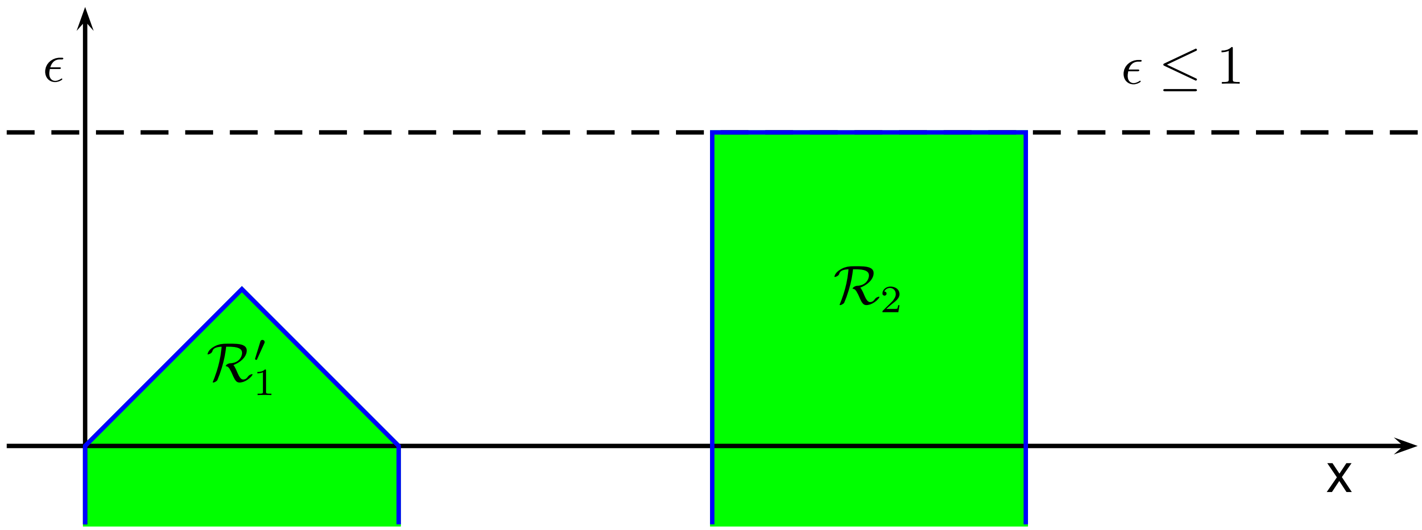
$\mathcal{R}_1 \uplus \mathcal{R}_2$ encodes $\mathcal{P}'' = \operatorname{con}\bigl(\{0 < x < 4\}\bigr)$.



. . . But This Time There is a Workaround!

In the encoding, for each strict inequality constraint, do also add the corresponding non-strict inequality.

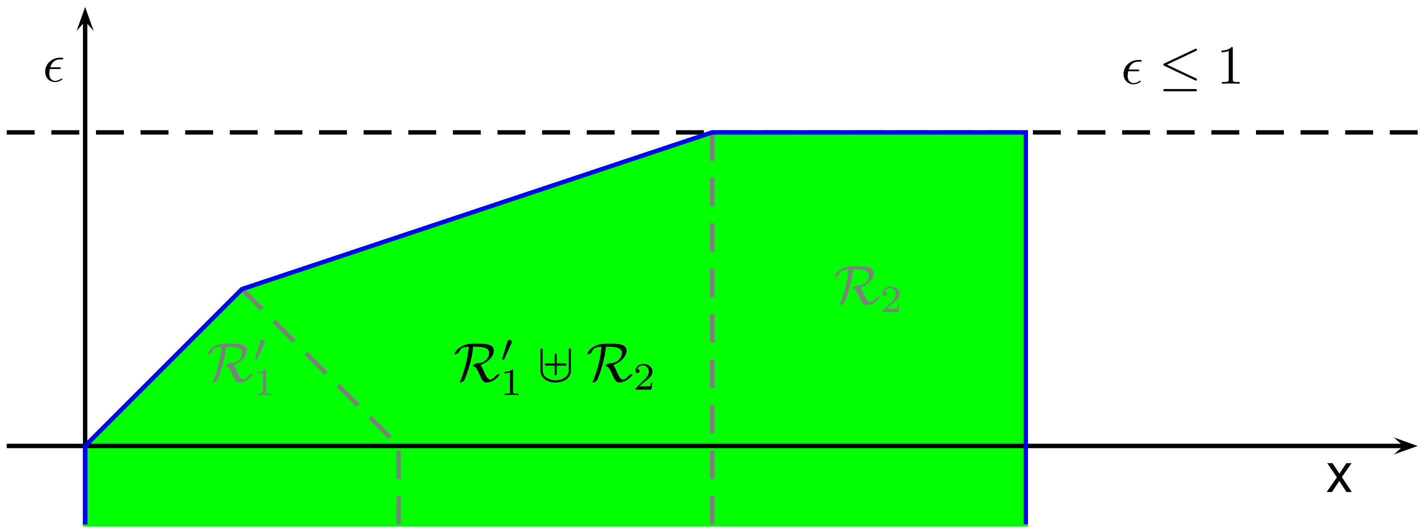
$$\mathcal{R}_1' \stackrel{\text{def}}{=} \text{con}\big(\{\epsilon \leq 1, x - \epsilon \geq 0, x \geq 0, -x - \epsilon \geq -1, -x \geq -1\}\big).$$



. . . But This Time There is a Workaround!

In the encoding, for each strict inequality constraint, do also add the corresponding non-strict inequality.

$$\mathcal{R}'_1 \stackrel{\text{def}}{=} \text{con}\big(\{\epsilon \leq 1, x - \epsilon \geq 0, x \geq 0, -x - \epsilon \geq -1, -x \geq -1\}\big).$$



THE ALTERNATIVE ENCODING

- If $\mathcal{P} \in \mathbb{P}_n$ and $\mathcal{P} = \mathrm{con}(\mathcal{C})$, where

$$\mathcal{C} = \left\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \bowtie_i b_i \;\middle|\; i \in \{1, \ldots, m\}, \boldsymbol{a}_i \in \mathbb{R}^n, \bowtie_i \in \{\geq, >\}, b_i \in \mathbb{R} \right\},$$

then $\mathcal{R} \in \mathbb{CP}_{n+1}$ is defined by $\mathcal{R} = \mathrm{con}\big(\mathrm{con\_repr}(\mathcal{C})\big)$, where

$$\begin{aligned}\mathrm{con\_repr}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \{\epsilon \leq 1\} & \\ & \cup \left\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle - 1 \cdot \epsilon \geq b_i \;\middle|\; i \in \{1, \ldots, m\}, \bowtie_i \in \{>\} \right\} \\ & \cup \left\{ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b_i \;\middle|\; i \in \{1, \ldots, m\}, \bowtie_i \in \{\geq, >\} \right\}.\end{aligned}$$

- If $\mathcal{P} \in \mathbb{P}_n$ and $\mathcal{P} = \mathrm{gen}(\mathcal{G})$, where $\mathcal{G} = (R, P, C)$, then $\mathcal{R} \in \mathbb{CP}_{n+1}$ is defined by $\mathcal{R} = \mathrm{gen}\big(\mathrm{gen\_repr}(\mathcal{G})\big) = \mathrm{gen}\big((R', P')\big)$, where

$$\begin{aligned} R' &= \left\{(\boldsymbol{0}^{\mathrm{T}}, -1)^{\mathrm{T}}\right\} \cup \left\{ (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\middle|\; \boldsymbol{r} \in R \right\}, \\ P' &= \left\{ (\boldsymbol{p}^{\mathrm{T}}, 1)^{\mathrm{T}} \;\middle|\; \boldsymbol{p} \in P \right\} \cup \left\{ (\boldsymbol{q}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\middle|\; \boldsymbol{q} \in C \right\}. \end{aligned}$$

CONSTRAINT-BIASED VS GENERATOR-BIASED REPRESENTATIONS

➜ The alternative encoding has dual properties with respect to the original by Halbwachs et al.

  ➜ With the original, the encoding of an NNC polyhedron may require a similar number of constraints but about twice the number of generators: it is *constraint-biased*.

  ➜ With the alternative, it may require a similar number of generators but twice the number of constraints: this encoding is *generator-biased*.

$\Longrightarrow$ Due to the use of exponential algorithms, their computational behavior can vary wildly depending on the operation and on the actual polyhedra being manipulated.

$\Longrightarrow$ The performance of one encoding with respect to the other will heavily depend on the particular application.

Minimization of ϵ-Polyhedra

- A minimized encoding may represent a non-minimized NNC polyhedron.
- In other words: in no way does minimization of the representation in $\mathbb{CP}_{n+1}$ imply minimization of the NNC polyhedron in $\mathbb{P}_n$.
  - this is true for both encodings.
- There are examples where a "minimized" representation has more than half of the constraints that are redundant.
- This causes both efficiency and usability problems:
  - performance can be severely limited by the presence of redundant constraints and generators;
  - the client application must distinguish between the real constraints/generators and the surrounding noise.

STRONG MINIMIZATION OF ϵ-POLYHEDRA

➜ A solution to this problem is presented in the paper:

Roberto Bagnara, Patricia M. Hill, and Enea Zaffanella

A New Encoding and Implementation of Not Necessarily Closed Convex Polyhedra

Quaderno 305, Department of Mathematics, University of Parma, 2002

Available at `http://www.cs.unipr.it/`

➜ There, we define a general notion of strong minimization that encompasses both the constraint- and the generator-biased encodings.

➜ Moreover, this notion of minimization maps constraint-biased representations to constraint-biased ones and likewise for the generator-biased representations.

➜ The contribution is important also from the practical point of view: the strong minimization procedure we propose is very efficient.

THE IMPACT OF STRONG MINIMIZATION

$\# P_i + \# C_i$	eval	Inters ($\# \mathcal{C}_i$)		Poly-hull ($\# \mathcal{G}_{ij}$)		Final result ($\# \mathcal{C}$)			
		1st arg	2nd arg	1st arg	2nd arg	res	smf	time	time-smf
4 + 8	a	48	48	131	77	356	**56**	0.91	0.01
	b	**32**	**32**	**40**	**17**	156	**56**	0.08	0.00
	c	48	**32**	132	**17**	251	**56**	0.16	0.00
8 + 8	a	62	62	209	125	537	**59**	2.29	0.01
	b	**36**	**36**	**50**	**21**	308	**59**	0.25	0.00
	c	62	**36**	190	**21**	332	**59**	0.37	0.00
8 + 10	a	132	132	414	305	2794	**227**	118.64	0.45
	b	**68**	**68**	**58**	**25**	1084	**227**	1.42	0.06
	c	132	**68**	261	**25**	1423	**227**	3.96	0.08
16 + 10	a	178	178	697	657	5078	**235**	932.72	2.07
	b	**80**	**80**	**78**	**29**	1775	**235**	5.24	0.14
	c	178	**80**	418	**29**	1238	**235**	9.48	0.08

Conclusion

- Convex polyhedra provide the basis for several abstractions used in static analysis and computer-aided verification of complex system.
- Some of these applications require the manipulation of convex polyhedra that are not necessarily closed.
- We have proposed an elegant way of decoupling the essential geometric features of NNC polyhedra from their implementation.
- This separation, besides providing a natural and easy to use interface, enables the search for new implementation techniques.
- We have shown that the standard implementation of NNC polyhedra, which happens to be biased for constraint-intensive computations, has a dual that is biased for generator-intensive computations.
- We have implemented all these ideas in the *Parma Polyhedra Library*

`http://www.cs.unipr.it/ppl/`