Precise Widening Operators for Convex Polyhedra

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- ➜ But some applications need more precision. Solutions include:
  1. the widening delay technique (Cousot, '81);
  2. the widening 'up to' technique (Halbwachs, CAV'93);
  3. various extrapolation operators (no convergence guarantee).

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- ➜ But some applications need more precision. Solutions include:
  - ① the widening delay technique (Cousot, '81);
  - ② the widening 'up to' technique (Halbwachs, CAV'93);
  - ③ various extrapolation operators (no convergence guarantee).
- ➜ Our goal: *provide a framework for the definition of new widening operators on convex polyhedra improving upon the precision of the standard widening.*

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➜ *Real widenings do provide a convergence guarantee.*

➜ Operators not doing so are better called extrapolation operators.

Definition of Widening Operator

A variant of the classical one (see Cousot and Cousot, PLILP'92):

- The operator $\nabla : L \times L \rightarrowtail L$ is a widening if
  1. $\forall x, y \in L : x \sqsubseteq y \implies y \sqsubseteq x \nabla y$;
  2. for all increasing chains $y_0 \sqsubseteq y_1 \sqsubseteq \cdots$, the increasing chain defined by $x_0 \stackrel{\text{def}}{=} y_0, \ldots, x_{i+1} \stackrel{\text{def}}{=} x_i \nabla y_{i+1}, \ldots$ is not strictly increasing.

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- ➜ The upward iteration sequence with widenings (starting from $x_0 \in L$)

$$x_{i+1} = \begin{cases} x_i, & \text{if } \mathcal{F}(x_i) \sqsubseteq x_i; \\ x_i \nabla \big(x_i \sqcup \mathcal{F}(x_i)\big), & \text{otherwise;} \end{cases}$$

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  converges after a finite number of iterations.
- Note: ∇ always applied to arguments $x = x_i$ and $y = x_i \sqcup \mathcal{F}(x_i)$ satisfying $x \sqsubseteq y$ and $x \neq y$.

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A lattice with respect to subset inclusion, with infinite chains.

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Constraint Representation: $\mathcal{P} = \mathrm{con}(\mathcal{C})$

- $\mathcal{C}$ is a finite set of linear non-strict inequality (resp., equality) constraints.
- No redundant constraint $+$ max number of equalities $\Longrightarrow$ minimal form.
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Generator Representation: $\mathcal{P} = \mathrm{gen}(\mathcal{G})$

- $\mathcal{G} = (L, R, P)$, where
  - P is a finite set of points of $\mathcal{P}$;
  - R is a finite set of rays (directions of infinity) of $\mathcal{P}$;
  - L is a finite set of lines (bidirectional rays) of $\mathcal{P}$.
- No redundant generator $+$ max number of lines $\implies$ minimal form.
- Points and rays orthogonal wrt lines $\implies$ orthogonal form.

The Standard Widening ∇

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- Intuitively, $\mathcal{P}_1 \nabla \mathcal{P}_2$ *is defined by all the constraints of* $\mathcal{P}_1 = \mathrm{con}(\mathcal{C}_1)$ *that are also satisfied by* $\mathcal{P}_2$*.*

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- The resulting operator is both precise and efficient: this "*tentative*" definition has been the one and only available approach for 25 years.
- Can we improve its precision? (Perhaps, trading some efficiency.)

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  where $\mathcal{C}_i$ is in minimal form and $\mathcal{G}_i = (L_i, R_i, P_i)$ is in orthogonal form;
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  ① $\dim(\mathcal{P}_1) < \dim(\mathcal{P}_2)$;
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  ③ $\# \mathcal{C}_1 > \# \mathcal{C}_2$;
  ④ $\# \mathcal{C}_1 = \# \mathcal{C}_2 \wedge \# P_1 > \# P_2$;
  ⑤ $\# \mathcal{C}_1 = \# \mathcal{C}_2 \wedge \# P_1 = \# P_2 \wedge \kappa(R_1) \gg \kappa(R_2)$.

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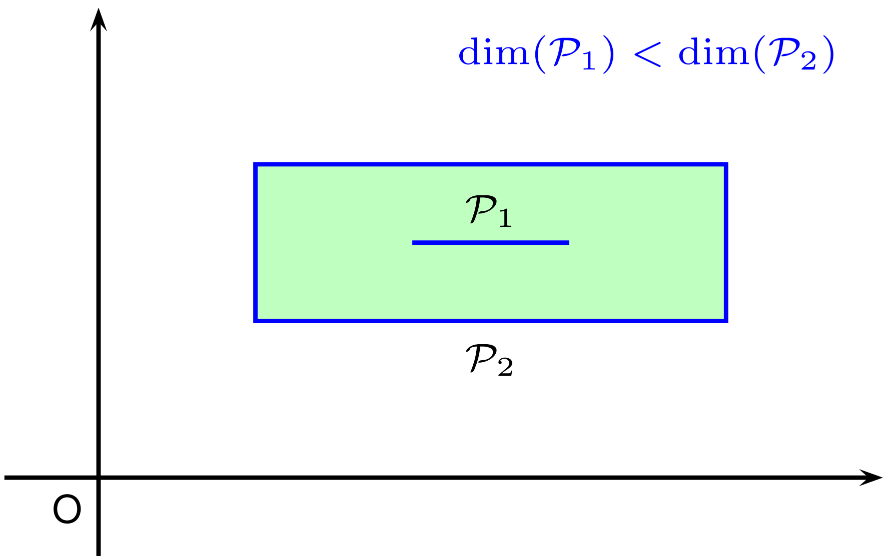
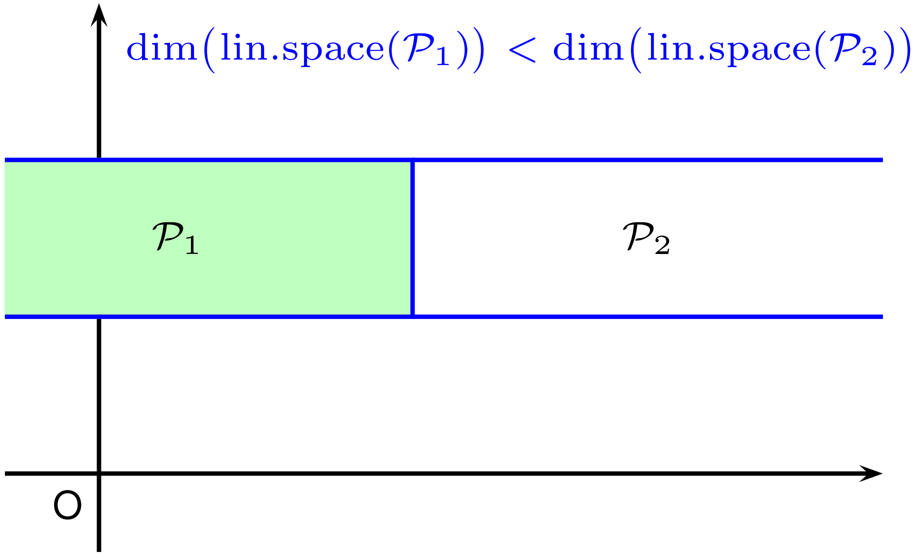
- Relation $\curvearrowright$ satisfies the ascending chain condition on $\mathbb{CP}_n$.

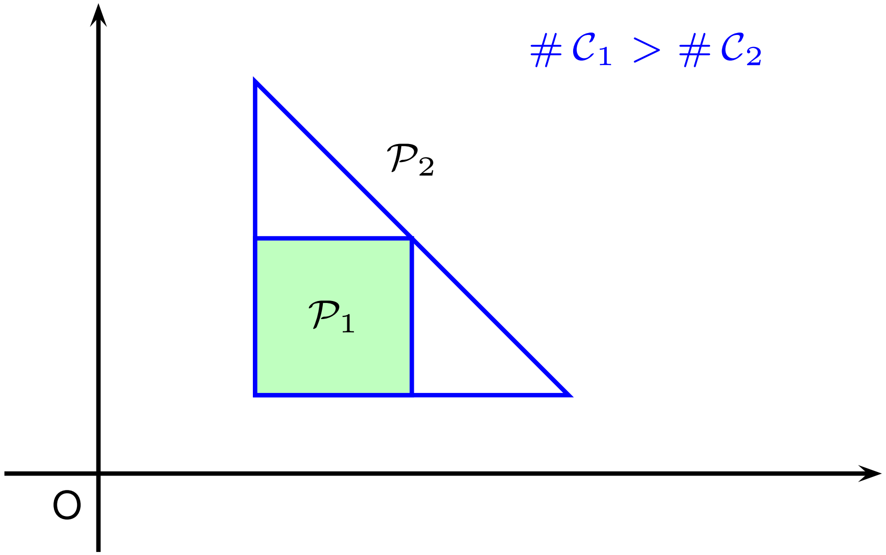
EXAMPLES FOR $\mathcal{P}_1 \curvearrowright \mathcal{P}_2$: CASE 1



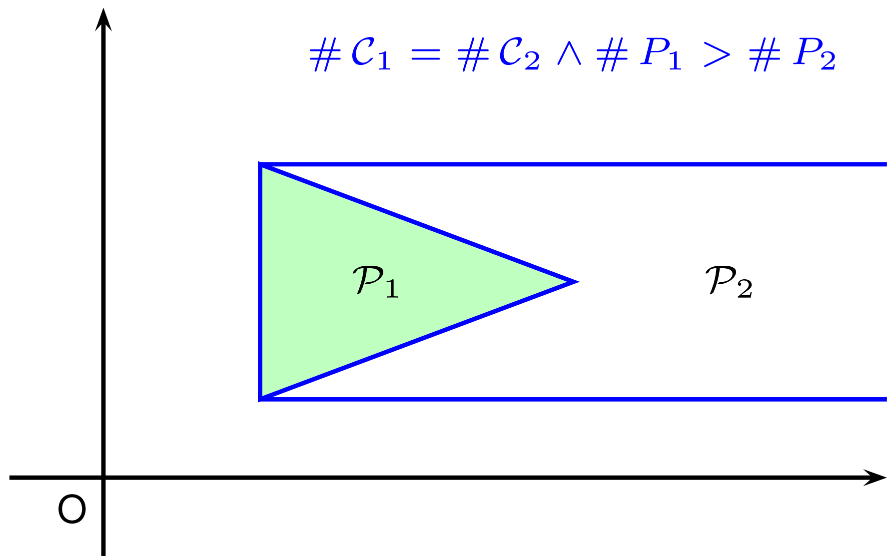
EXAMPLES FOR $\mathcal{P}_1 \curvearrowright \mathcal{P}_2$: CASE 2



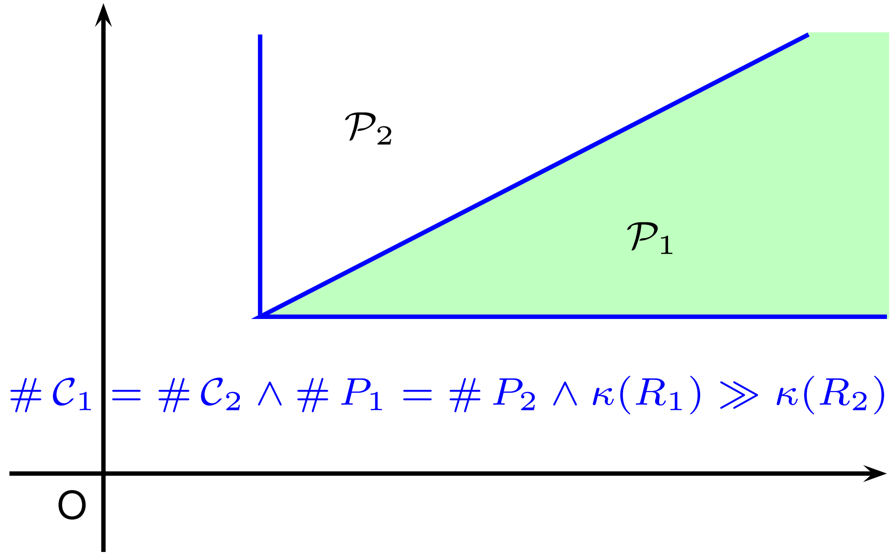
EXAMPLES FOR $\mathcal{P}_1 \curvearrowright \mathcal{P}_2$: CASE 3



EXAMPLES FOR $\mathcal{P}_1 \curvearrowright \mathcal{P}_2$: CASE 4



EXAMPLES FOR $\mathcal{P}_1 \curvearrowright \mathcal{P}_2$: CASE 5



A Framework for Defining New Widenings

The key results.

- ➜ The standard widening satisfies $\mathcal{P}_1 \curvearrowright \mathcal{P}_1 \nabla \mathcal{P}_2$.
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- For any upper bound operator $h \colon \mathbb{CP}_n \times \mathbb{CP}_n \to \mathbb{CP}_n$, define

$$\mathcal{P}_1 \tilde{\nabla} \mathcal{P}_2 \stackrel{\text{def}}{=} \begin{cases} h(\mathcal{P}_1, \mathcal{P}_2), & \text{if } \mathcal{P}_1 \curvearrowright h(\mathcal{P}_1, \mathcal{P}_2) \subset \mathcal{P}_1 \nabla \mathcal{P}_2; \\ \mathcal{P}_1 \nabla \mathcal{P}_2, & \text{otherwise.} \end{cases}$$

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Then:

① $\tilde{\nabla}$ is a widening operator;

② $\tilde{\nabla}$ is at least as precise as the standard widening.

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- ➜ No precision loss: to be tried before all other techniques.
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- Applicable whenever $\mathcal{P}_1 \curvearrowright \mathcal{P}_2$.
- No precision loss: to be tried before all other techniques.
- Already suggested by Cousot and Cousot, PLILP'92.
- All the other techniques may safely assume $\mathcal{P}_1 \not\curvearrowright \mathcal{P}_2$.
- Since by hypothesis $\mathcal{P}_1 \subset \mathcal{P}_2$, we can also assume

$$\begin{aligned} \mathrm{aff.hull}(\mathcal{P}_1) &= \mathrm{aff.hull}(\mathcal{P}_2), \\ \mathrm{lin.space}(\mathcal{P}_1) &= \mathrm{lin.space}(\mathcal{P}_2). \end{aligned}$$

2ND HEURISTICS: COMBINING CONSTRAINTS

Let $h_c(\mathcal{P}_1, \mathcal{P}_2) \stackrel{\text{def}}{=} \mathrm{con}(\mathcal{C}_\oplus) \cap (\mathcal{P}_1 \nabla \mathcal{P}_2)$, where

- ➜ $\mathcal{C}_\nabla$ are the constraints of the standard widening;
- ➜ $\mathcal{C}_\oplus \stackrel{\text{def}}{=} \left\{ \oplus(\mathcal{C}_{\boldsymbol{p}}) \;\middle|\; \begin{array}{l} \boldsymbol{p} \in P_1, \mathrm{sat\_con}\big(\boldsymbol{p}, \mathrm{ineq}(\mathcal{C}_\nabla)\big) = \varnothing, \\ \mathcal{C}_{\boldsymbol{p}} = \mathrm{sat\_con}\big(\boldsymbol{p}, \mathrm{ineq}(\mathcal{C}_2)\big) \neq \varnothing \end{array} \right\}$.
- ➜ $\oplus$ is a (deliberately left unspecified) convex combination.

Informally, *we ensure that each non-redundant point $\boldsymbol{p} \in \mathcal{P}_1$ that was lying on a facet of $\mathcal{P}_2$ will still lie on a facet of $h_c(\mathcal{P}_1, \mathcal{P}_2)$.*

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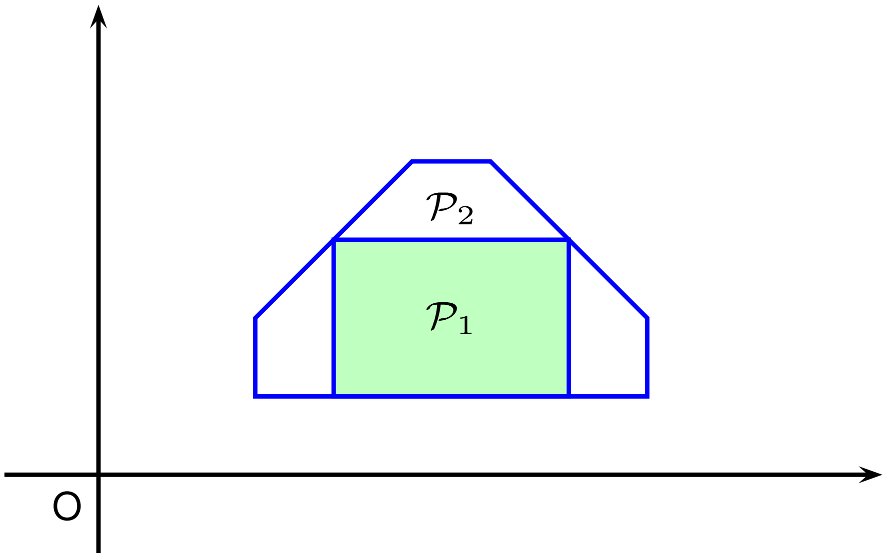
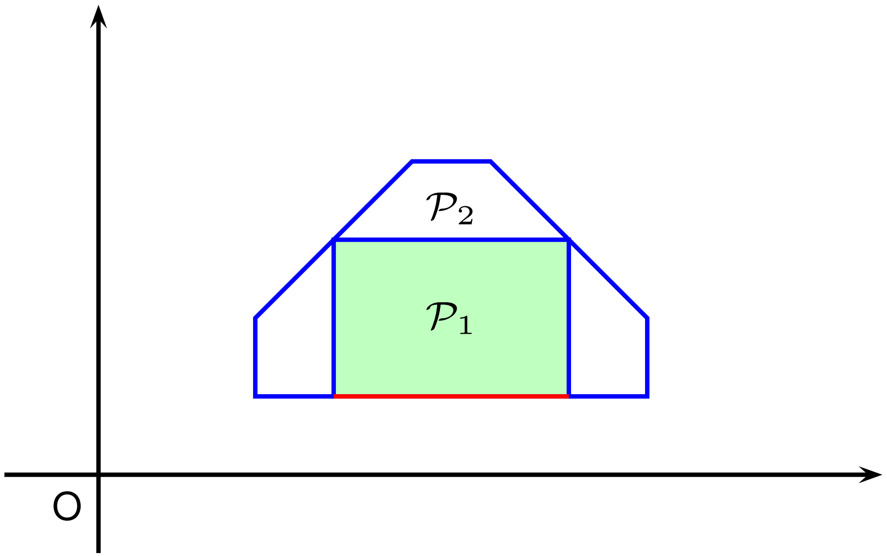
- ➜ Besson et al., SAS'99 suggest to average the constraints in $\mathcal{C}_{\boldsymbol{p}}$.
- ➜ Afterall, the choice of $\oplus$ is arbitrary: we opted for a simpler combination.
- ➜ A similar heuristics, with no convergence guarantee, was proposed by Henzinger et al., CDC'01.

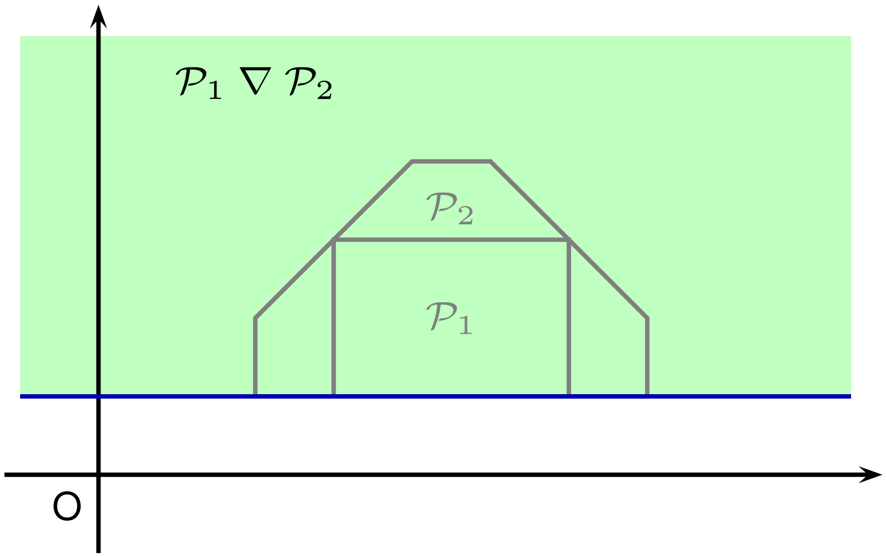
Standard Widening vs. Combining Constraints (i)



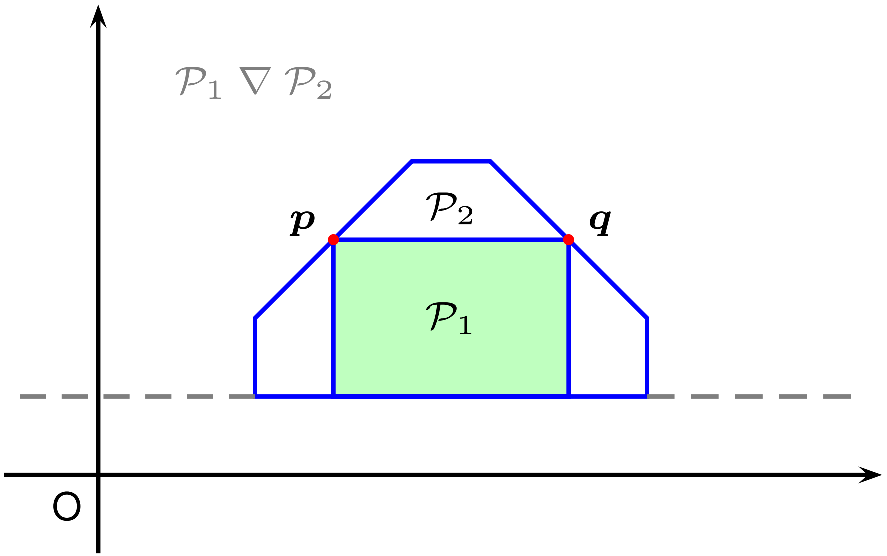
STANDARD WIDENING VS. COMBINING CONSTRAINTS (II)



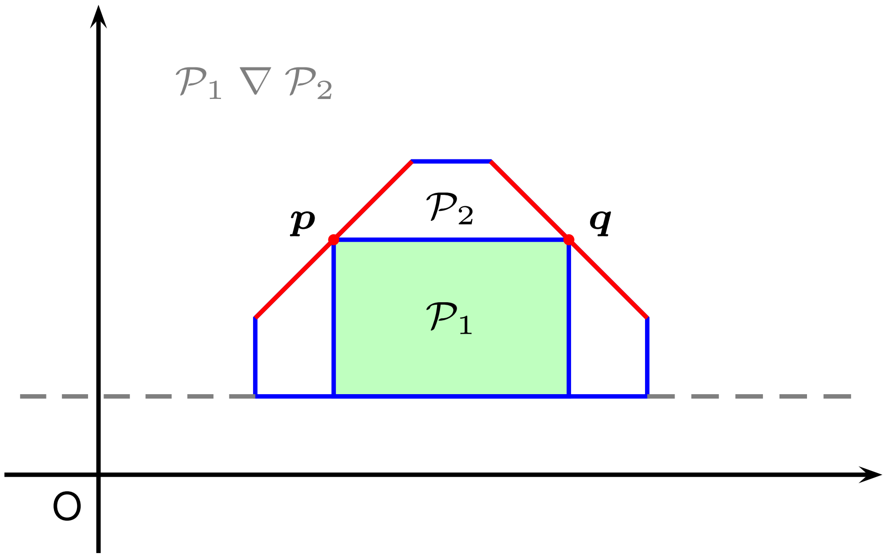
Standard Widening vs. Combining Constraints (iii)



Standard Widening vs. Combining Constraints (iv)



STANDARD WIDENING VS. COMBINING CONSTRAINTS (V)



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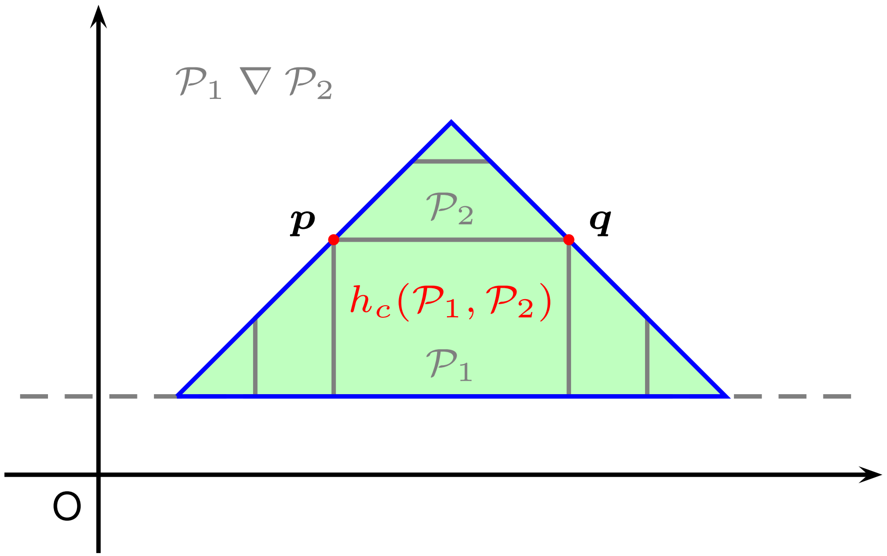

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➜ Consider the set of rays

$$R \stackrel{\mathrm{def}}{=} \left\{ \boldsymbol{p}_2 - \boldsymbol{p}_1 \;\middle|\; \boldsymbol{p}_1 \in P_1, \boldsymbol{p}_2 \in P_2 \setminus P_1 \right\}.$$

➜ Informally, *each point $\boldsymbol{p}_2 \in P_2 \setminus P_1$ is seen as an evolution of point $\boldsymbol{p}_1 \in P_1$. By generating the ray $\boldsymbol{p}_2 - \boldsymbol{p}_1$, we extrapolate this evolution towards infinity.*

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- Thus, let $h_p(\mathcal{P}_1, \mathcal{P}_2) \stackrel{\text{def}}{=} \mathrm{gen}\big((L_2, R_2 \cup R, P_2)\big) \cap (\mathcal{P}_1 \nabla \mathcal{P}_2)$.

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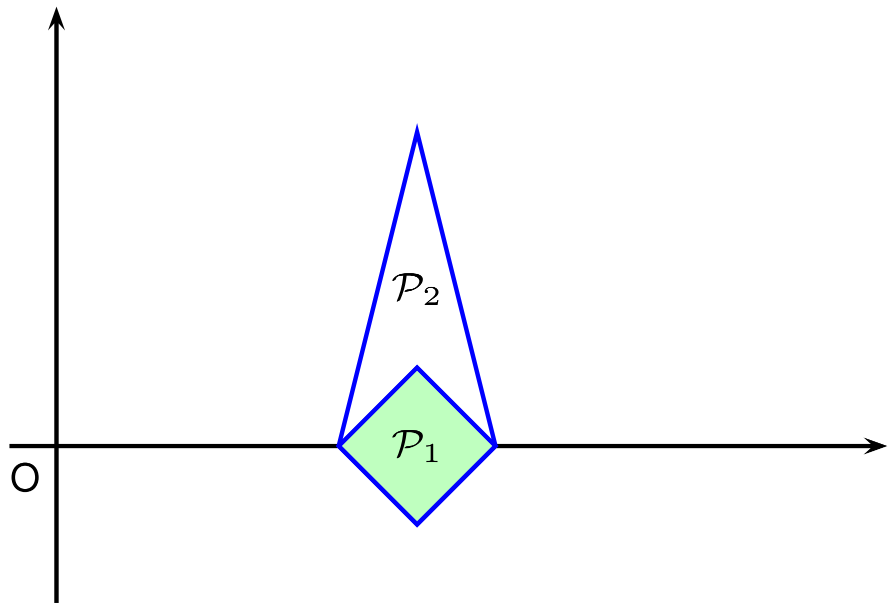

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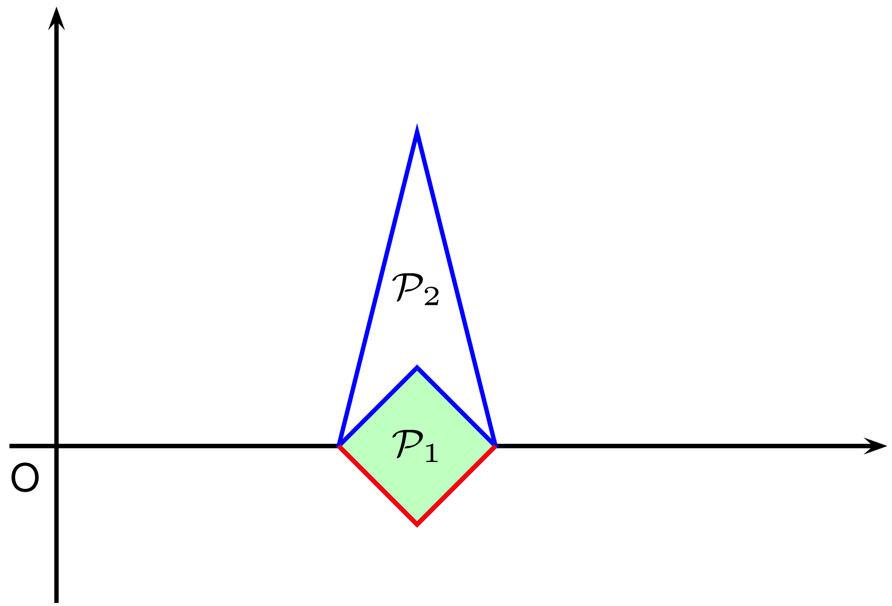

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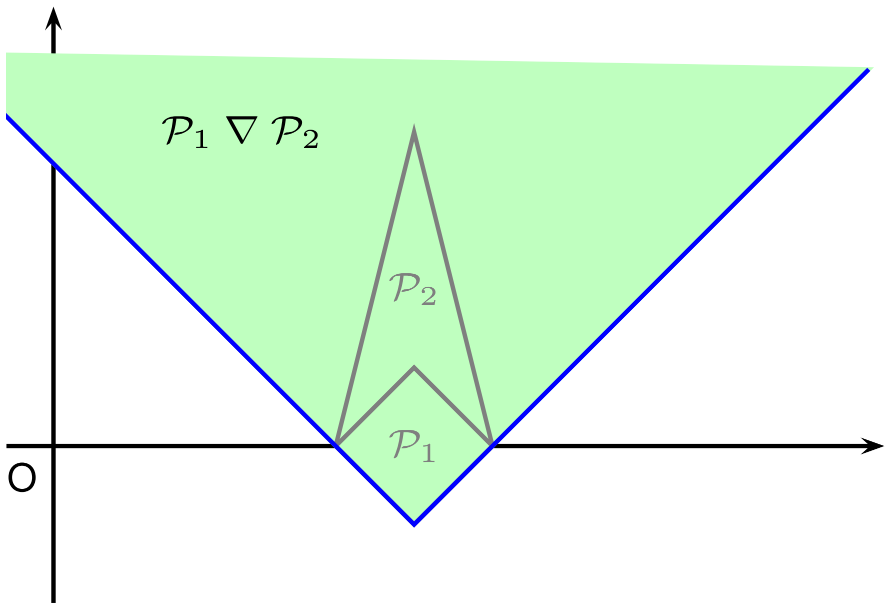

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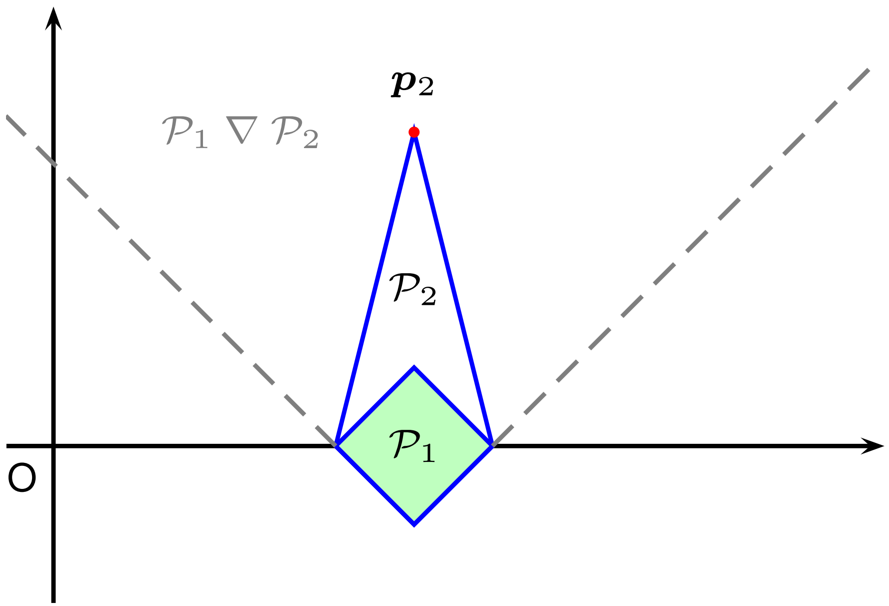
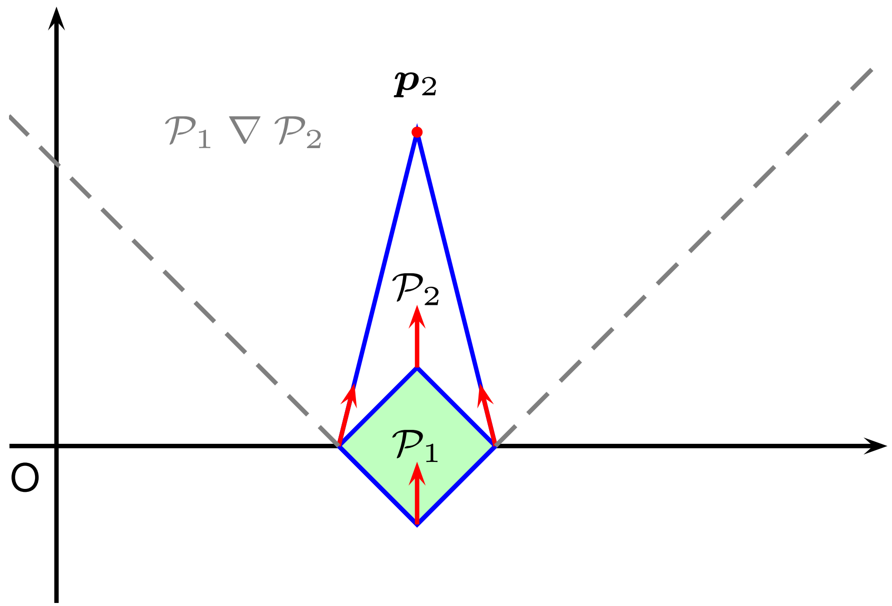
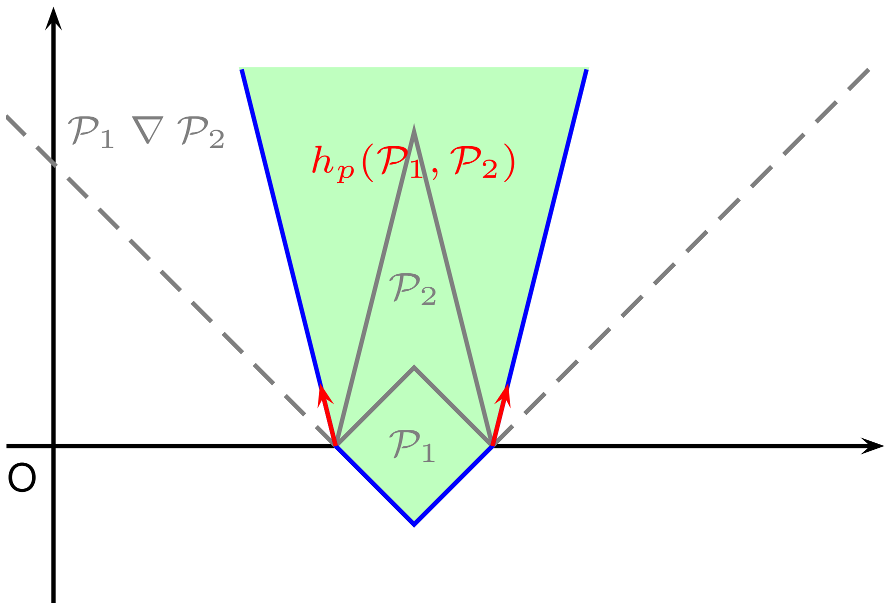


Standard Widening vs. Evolving Points (v)



Standard Widening vs. Evolving Points (vi)



4TH HEURISTICS: EVOLVING RAYS

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➜ Informally, *each ray $\boldsymbol{r}_2 \in R_2 \setminus R_1$ is seen as an evolution of ray $\boldsymbol{r}_1 \in R_1$. We extrapolate this evolution by rotating ray $\boldsymbol{r}_2$, stopping as soon as it touches the boundary of the Cartesian orthant*.

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- ➜ Thus, let $h_r(\mathcal{P}_1, \mathcal{P}_2) \stackrel{\text{def}}{=} \mathrm{gen}\bigl((L_2, R_2 \cup R, P_2)\bigr) \cap (\mathcal{P}_1 \nabla \mathcal{P}_2)$.

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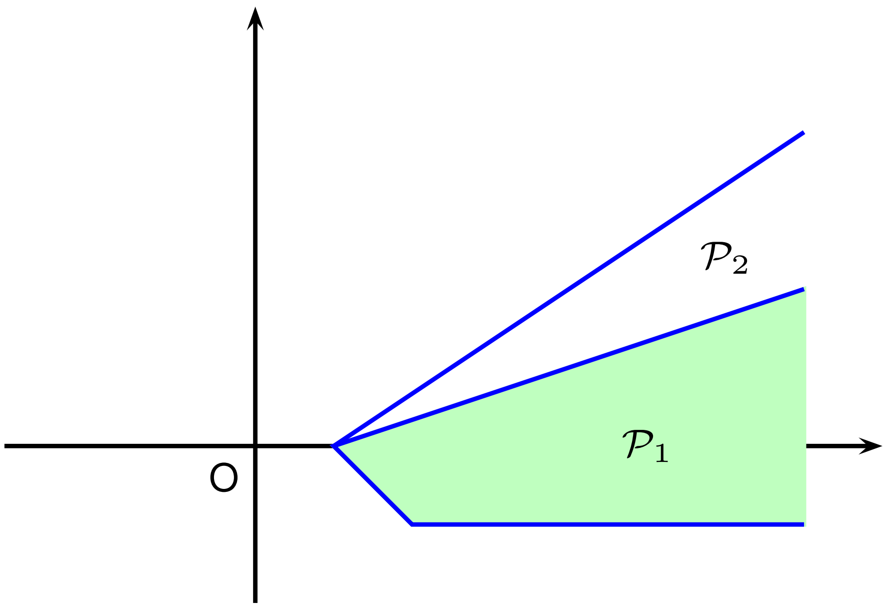
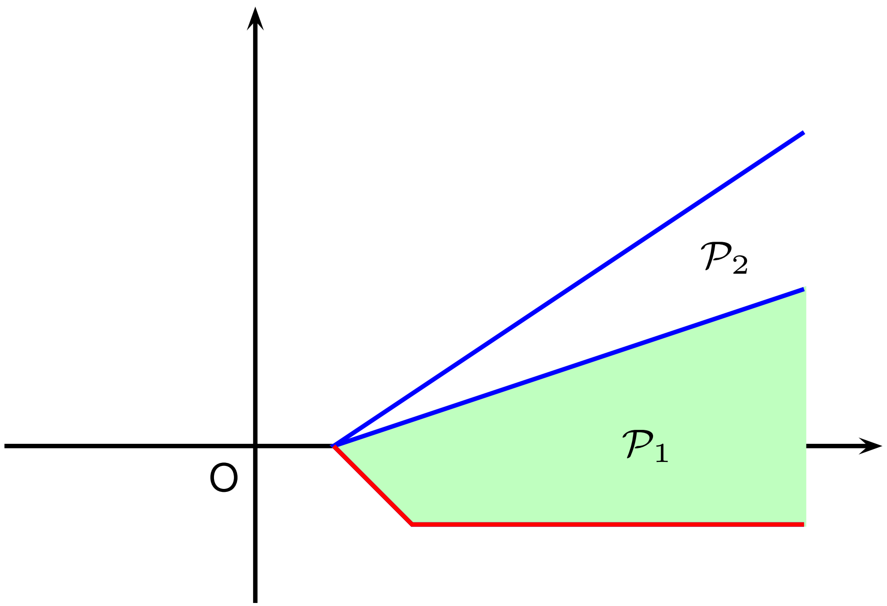
- The extrapolation will decrease the total number of non-zero coordinates of the ray $\Longrightarrow$ hopefully satisfying the last case in the definition of the limited growth ordering $\curvearrowright$:

$$\#\mathcal{C}_1 = \#\mathcal{C}_2 \wedge \# P_1 = \# P_2 \wedge \kappa(R_1) \gg \kappa(R_2).$$

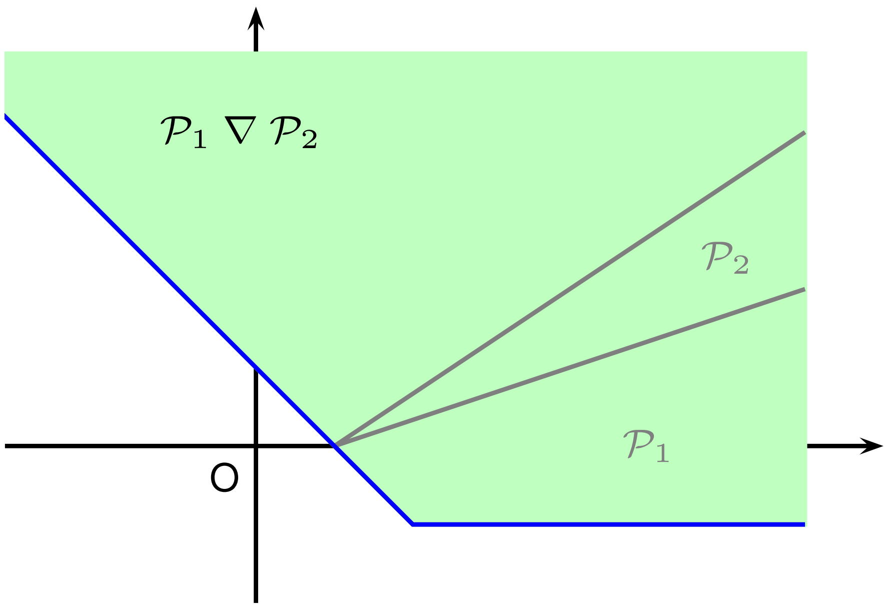
Standard Widening vs. Evolving Rays (i)



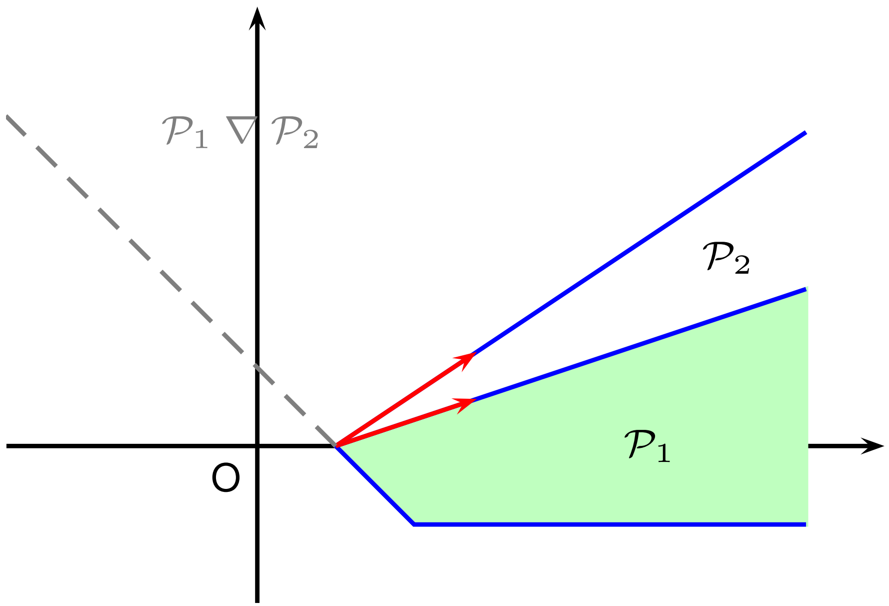
Standard Widening vs. Evolving Rays (ii)



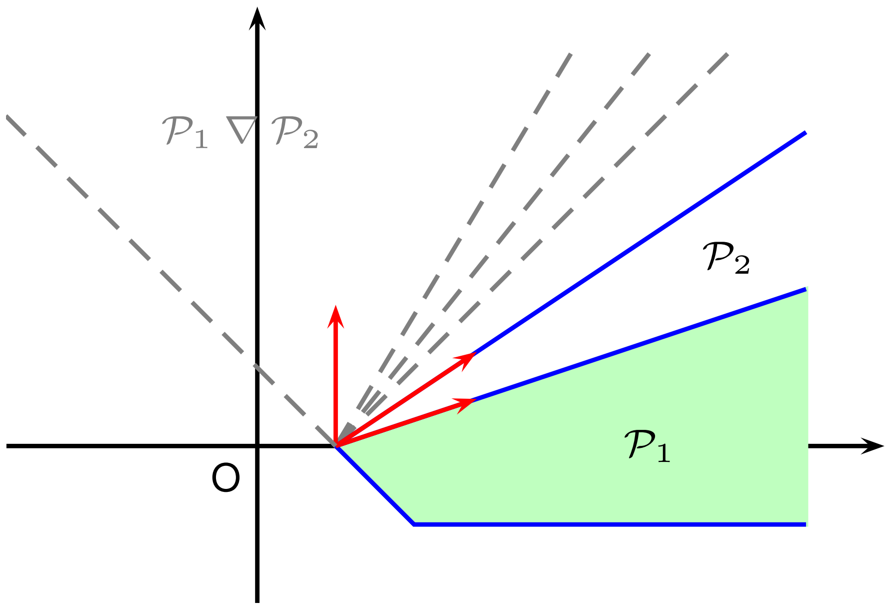
Standard Widening vs. Evolving Rays (iii)



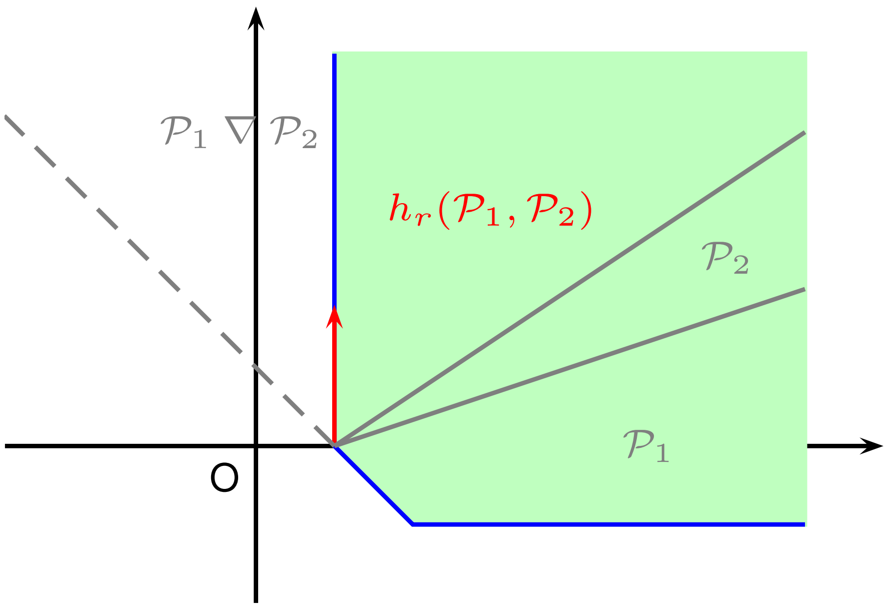
Standard Widening vs. Evolving Rays (iv)



Standard Widening vs. Evolving Rays (v)



Standard Widening vs. Evolving Rays (vi)



The New Widening $\hat{\nabla}$

➜ An instance of the framework: try the four heuristics in the given order, eventually falling back to the standard widening.

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- In general, this does not hold for the final result of upward iteration sequences, because neither the standard widening nor the new one are monotonic operators.

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Note: carefully chosen widening strategy (Bourdoncle, FMPTA'93)
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	# programs (361)			# predicates (23279)		
k (delay)	improve	degr	incomp	improve	degr	incomp
0	121	-	2	1340	3	2
1	34	-	-	273	-	-
2	29	-	-	222	-	-
3	28	-	-	160	-	-
4	25	-	2	126	2	-
10	25	-	-	124	-	-

Efficiency Comparison

Argument size relations for Prolog programs using China $+$ PPL.

Total analysis time

	std ∇_k		new $\hat{\nabla}_k$	
k (delay)	all	top 20	all	top 20
0	**1.00**	0.72	1.05	0.77
1	1.09	0.79	1.11	0.80
2	1.16	0.83	1.18	0.84
3	1.23	0.88	1.25	0.89
4	1.32	0.95	1.34	0.95
10	1.82	1.23	1.85	1.24

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The PPL is free software: everything is available at

`http://www.cs.unipr.it/ppl/`