The Parma Polyhedra Library User's Manual[*] (version 0.1)

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October 23, 2001

[*]This work has been partly supported by: University of Parma's FIL scientific research project (ex 60%) "Pure and Applied Mathematics"; MURST project "Automatic Program Certification by Abstract Interpretation"; MURST project "Abstract Interpretation, Type Systems and Control-Flow Analysis".

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This document describes the Parma Polyhedra Library (PPL).

For the most up-to-date information see the Parma Polyhedra Library WWW site:

http://www.cs.unipr.it/ppl/

Contents

1 Convex Polyhedra and the PPL

1.1 An Introduction to Convex Polyhedra

The following definitions and results are taken from:

- G. L. Nemhauser and L. A. Wolsey - Integer and Combinatorial Optimization - Wiley Interscience Series in Discrete Mathematics and Optimization, 1988.
- D. K. Wilde - A library for doing polyhedral operations - IRISA Publication interne n. 785, December 1993.

- K. Fukuda - Polyhedral Computation FAQ - Swiss Federal Institute of Technology, Lausanne and Zurich, Switzerland, October 2000.

Combination

Let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$. The linear combination $\sum_{j=1}^{k} \lambda_j \mathbf{x}_j$ is said to be

- a *positive combination*, if $\forall j \in \{1, \dots, k\} : \lambda_j \geq 0$;
- an *affine combination*, if $\sum_{j=1}^{k} \lambda_j = 1$;
- a *convex combination*, if both the previous conditions hold.

Note that when $k = 0$, $\sum_{j=1}^{k} \lambda_j \mathbf{x}_j = \mathbf{0}$ and $\sum_{j=1}^{k} \lambda_j = 0$. This means that $\sum_{j=1}^{0} \lambda_j \mathbf{x}_j$ may be regarded as a positive but not an affine combination.

Scalar product

Let $\mathbf{x} = (x_0, \dots, x_{n-1})^{\mathrm{T}}$, $\mathbf{y} = (y_0, \dots, y_{n-1})^{\mathrm{T}} \in \mathbb{R}^n$. The *scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=0}^{n-1} x_i y_i.$$

The vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Convex hull

The *convex hull* of a set $K \subseteq \mathbb{R}^n$ is the set of all the convex combinations of the points in K. The set K is convex if it is its own convex hull.

Affine transformation

An *affine transformation* is a function mapping a point $\mathbf{x} \in \mathbb{R}^n$ to a point $\mathbf{x}' \in \mathbb{R}^m$ such that

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}$$

where $A \in \mathbb{R}^m \times \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$.

Linear independence

A set of points $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$ is *linearly independent* if, for all $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, the set of equations

$$\sum_{i=1}^{k} \lambda_i \mathbf{x}_i = \mathbf{0}$$

implies that, for each $i = 1, \dots, k$, $\lambda_i = 0$.

Note that the maximum number of linearly independent points in $\mathbb{R}^n$ is n.

Proposition

If A is an $m \times n$ matrix, the maximum number of linearly independent rows of A, viewed as vectors of $\mathbb{R}^n$, equals the maximum number of linearly independent columns of A, viewed as vectors of $\mathbb{R}^m$.

Rank

The maximum number of linearly independent rows (columns) of a matrix A is the *rank* of A and is denoted by $\mathrm{rank}(A)$.

Affine independence

A set of points $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$ is *affinely independent* if, for all $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, the set of equations

$$\sum_{i=1}^{k} \lambda_i \mathbf{x}_i = \mathbf{0}, \qquad \sum_{i=1}^{k} \lambda_i = 0$$

implies that, for each $i = 1, \dots, k$, $\lambda_i = 0$.

Note that linear independence implies affine independence, but the converse is not true. Moreover the maximum number of affinely independent points in $\mathbb{R}^n$ is $n+1$ (e.g., n linearly independent points and the origin $\mathbf{0}$).

Polyhedron

A set $P \subseteq \mathbb{R}^n$ is called a *polyhedron* if it is the set of solutions to a finite number of linear equalities and inequalities:

$$P = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}, C\mathbf{x} \geq \mathbf{d} \,\},$$

where, if m_1 is the number of linear equalities and m_2 the number of linear inequalities, $A \in \mathbb{R}^{m_1} \times \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^{m_1}$, $C \in \mathbb{R}^{m_2} \times \mathbb{R}^n$ and $\mathbf{d} \in \mathbb{R}^{m_2}$.

In the sequel, we will simply write "equality" and "inequality" to mean "linear equality" and "linear inequality", respectively; also, we will refer to either an equality or an inequality as a *constraint*.

Constraints representation

It follows that a polyhedron $P \subseteq \mathbb{R}^n$ can be always represented by a *system of constraints*:

$$P = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \,\}$$

for some matrix $A \in \mathbb{R}^m \times \mathbb{R}^n$ and vector $\mathbf{b} \in \mathbb{R}^m$.

Note that, if $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ and the system of constraints contains the two inequalities $\langle \mathbf{c}, \mathbf{x} \rangle \geq \lambda$ and $\langle \mathbf{c}, \mathbf{x} \rangle \leq \lambda$ (i.e., $\langle -\mathbf{c}, \mathbf{x} \rangle \geq -\lambda$), then they can be replaced by the equivalent unique *equality* $\langle \mathbf{c}, \mathbf{x} \rangle = \lambda$. Conversely, if we have an equality, then it can be replaced by two inequalities (as above).

Rational polyhedron

A polyhedron $P \subseteq \mathbb{R}^n$ is said to be *rational* if there exists a matrix $A \in \mathbb{R}^{m'} \times \mathbb{R}^n$ and a vector $\mathbf{b} \in \mathbb{R}^{m'}$ with rational coefficients such that

$$P = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \,\}.$$

In the sequel, we will consider only rational polyhedra and assume that, if $\{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \,\}$ is a system of constraints representing a polyhedron, then A and $\mathbf{b}$ have rational coefficients.

Universe polyhedron

A polyhedron $P \subseteq \mathbb{R}^n$ is called *universe polyhedron* if it is the whole space (i.e. $P = \mathbb{R}^n$).

Polytope

A polyhedron $P \subset \mathbb{R}^n$ is *bounded* if there exists a $\lambda \in \mathbb{R}, \lambda > 0$ such that

$$P \subseteq \left\{\, (x_0, \dots, x_{n-1})^{\mathrm{T}} \in \mathbb{R}^n \;\middle|\; -\lambda \leq x_j \leq \lambda \text{ for } j = 0, \dots, n-1 \,\right\}.$$

A bounded polyhedron is called a *polytope*.

Proposition

A polyhedron is a closed convex set.

Dimension

A polyhedron $P \subseteq \mathbb{R}^n$ is of *dimension* k, denoted by $\dim(P) = k$, if the maximum number of affinely independent points in P is $k+1$.

Vertex

A *vertex* of a polyhedron P is any point in P which cannot be expressed as a convex combination of any other distinct points in P.

Ray

Let P, P_0 be the polyhedra

$$P = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \,\} \neq \varnothing \quad \text{and } P_0 = \{\, \mathbf{r} \in \mathbb{R}^n \mid A\mathbf{r} \geq \mathbf{0} \,\}$$

where $A \in \mathbb{R}^m \times \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$. Then any point $\mathbf{r} \in P_0 \setminus \{\mathbf{0}\}$ is called a *ray* of P.

A ray indicates a direction in which the polyhedron P is infinite (i.e., unbounded).

Proposition

A point $\mathbf{r} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is a ray of a non-empty polyhedron $P \subseteq \mathbb{R}^n$ if and only if, for any point $\mathbf{x} \in P$, $(\mathbf{x} + \mu\mathbf{r}) \in P$ for all $\mu \in \mathbb{R}, \mu > 0$.

Extreme ray

A ray $\mathbf{r}$ of a polyhedron P is an *extreme ray* if there do not exist two rays $\mathbf{r}_1$ and $\mathbf{r}_2$ of P, where $\mathbf{r}_1 \neq \lambda\mathbf{r}_2$ for any $\lambda \in \mathbb{R}, \lambda > 0$, such that

$$\mathbf{r} = \mu_1\mathbf{r}_1 + \mu_2\mathbf{r}_2,$$

where $\mu_1, \mu_2 \in \mathbb{R}$, $\mu_1 > 0$ and $\mu_2 > 0$.

Line

A *line* (or *bidirectional ray*) of a polyhedron $P \subseteq \mathbb{R}^n$ is a ray $\mathbf{l}$ of P such that $-\mathbf{l}$ is another ray of P.

Cone

A set $C \subseteq \mathbb{R}^n$ is a *cone* if

$$\mathbf{x} \in C \Rightarrow \lambda\mathbf{x} \in C \text{ for all } \lambda \in \mathbb{R}, \lambda \geq 0.$$

Polyhedral cone

The polyhedron $P = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{0} \,\}$ is a convex cone and is called *polyhedral cone*.

Thus, a polyhedral cone is either *pointed*, having the origin as its only vertex, or has no vertices at all.

Lineality space

Given a polyhedron $P = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \,\}$, the *lineality space* of P is the set

$$\{\, \mathbf{x} \in P \mid A\mathbf{x} = \mathbf{0} \,\}$$

and it is denoted by $\operatorname{lin.space}(P)$.

Minkowski's sum

Let $R, S \subseteq \mathbb{R}^n$ be two sets of vectors. Then the *Minkowski's sum* of R and S is:

$$R + S = \{\, r + s \mid r \in R, s \in S \,\}.$$

Generators representation

A polyhedron $P \subseteq \mathbb{R}^n$ can also be represented by a finite set V of points of P, a finite set R of rays of P and a finite set L of lines of P. The elements of these three sets are the *generators* of P, in the sense that

$$P = \mathcal{V} + \mathcal{R} + \mathcal{L},$$

where the symbol '$+$' denotes the Minkowski's sum and

- $\mathcal{V}$ is the set of all the convex combinations of the points in V;
- $\mathcal{R}$ is the set of all the non-negative combinations of the rays in R; and
- $\mathcal{L}$ is the set of all the linear combinations of the lines in L.

Note that: $\mathcal{V}$ is a polytope, $\mathcal{R}$ is a pointed cone, and $\mathcal{L}$ is $\mathrm{lin.space}(P)$.

Note also that V must contain all vertices of P. However, V can contain other points, particularly if P is a non-empty polyhedron having no vertices (e.g., a half-space).

In the case that P contains at least one vertex, (in which case, $L = \emptyset$) the following two theorems justify this terminology.

Minkowski's theorem

Let $P = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \,\}$ be a non-empty polyhedron where $\mathrm{rank}(A) = n$. Let V be the set of vertices and R the set of extreme rays of P. Let also $\mathcal{V}$ be the set of convex combinations of V and $\mathcal{R}$ the set of positive combinations of R. Then

$$P = \mathcal{V} + \mathcal{R}.$$

The conditions that P is not empty and $\mathrm{rank}(A) = n$ required by this theorem are equivalent to the condition that P has a vertex. This condition is needed since, if the set of vertices $V = \emptyset$, then $\mathcal{V} = \emptyset$ and hence $\mathcal{V} + \mathcal{R}$ is also empty even though P may contain a line. (See also Nemhauser and Wolsey - Integer and Combinatorial Optimization - propositions 4.1 and 4.2 on pages 92 and 93).

The second theorem, called Weil's theorem, states that, starting from a system of generators (having rational coefficients), we can build a rational polyhedron:

Weil's theorem

If A is a rational $m \times n$ matrix, B is a rational $m' \times n$ matrix and

$$Q = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \mathbf{x}^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}} A + \mathbf{z}^{\mathrm{T}} B, \\ \mathbf{y} = (y_0, \ldots, y_{m-1})^{\mathrm{T}} \in \mathbb{R}^m_+, \sum_{k=0}^{m-1} y_k = 1, \\ \mathbf{z} \in \mathbb{R}^{m'}_+ \end{array} \right\}.$$

then Q is a rational polyhedron.

In fact, since Q consists of the sum of convex combinations of the rows of A with positive combinations of the rows of B, we can think of A as the matrix of vertices and B as the matrix of rays.

Dual representation

Thus a rational polyhedron P has a *dual representation*. That is, P can be represented by a system of constraints or a system of generators. Moreover, given one of the representations, there is an algorithm for computing the other.

(The following spurious string of characters in the user manual is due to a bug in Doxygen.)

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2 PPL Namespace Index

2.1 PPL Namespace List

Here is a list of all documented namespaces with brief descriptions:

3 PPL Hierarchical Index

3.1 PPL Class Hierarchy

This inheritance list is sorted roughly, but not completely, alphabetically:

4 PPL Compound Index

4.1 PPL Compound List

Here are the classes, structs, unions and interfaces with brief descriptions:

5 PPL Page Index

5.1 PPL Related Pages

Here is a list of all related documentation pages:

6 PPL Namespace Documentation

6.1 Parma_Polyhedra_Library Namespace Reference

The entire library is confined into this namespace.

Compounds

- class **Parma_Polyhedra_Library::Variable**

  A dimension of the space.

- class **Parma_Polyhedra_Library::LinExpression**

  A linear expression.

- class **Parma_Polyhedra_Library::Constraint**

  A linear equality or inequality.

- class **Parma_Polyhedra_Library::ConSys**

  A system of constraints.

- class **Parma_Polyhedra_Library::ConSys::const_iterator**
- class **Parma_Polyhedra_Library::Generator**

  A line, ray or vertex.

- class **Parma_Polyhedra_Library::GenSys**

  A system of generators.

- class **Parma_Polyhedra_Library::GenSys::const_iterator**
- class **Parma_Polyhedra_Library::Polyhedron**

  A convex polyhedron.

Non-friend operators on objects of the class Variable.

- std::ostream & **operator**<< (std::ostream &s, const **Parma_Polyhedra_Library::Variable** &v)

  Output operator.

- bool **operator**< (const **Variable** &v, const **Variable** &w)

Defines a total ordering on variables.

Non-friend operators on objects of the class Constraint.

- std::ostream & **operator**<< (std::ostream &s, const **Constraint** &c)

  Output operator.

Non-friend operators on objects of the class Generator.

- std::ostream & **operator**<< (std::ostream &s, const **Generator** &g)

  Output operator.

Non-friend operators on objects of the class Polyhedron.

- bool **operator==** (const **Polyhedron** &x, const **Polyhedron** &y)

  Returns `true` *if and only if* `x` *and* `y` *are the same polyhedron.*

- bool **operator!=** (const **Polyhedron** &x, const **Polyhedron** &y)

  Returns `true` *if and only if* `x` *and* `y` *are different polyhedra.*

- bool **operator**< (const **Polyhedron** &x, const **Polyhedron** &y)

  Returns `true` *if and only if* `x` *is strictly contained in* `y`.

- bool **operator**> (const **Polyhedron** &x, const **Polyhedron** &y)

  Returns `true` *if and only if* `x` *strictly contains* `y`.

- bool **operator**>**=** (const **Polyhedron** &x, const **Polyhedron** &y)

  Returns `true` *if and only if* `x` *contains* `y`.

Enumerations

- enum **GenSys_Con_Rel** { **NONE_SATISFIES**, **ALL_SATISFY**, **ALL_SATURATE**, **SOME_-SATISFY** }

  Describes possible relations between a system of generators and a given constraint.

6.1.1 Enumeration Type Documentation

6.1.1.1 enum Parma_Polyhedra_Library::GenSys_Con_Rel

Enumeration values:

NONE_SATISFIES No generator satisfies the given constraint.

ALL_SATISFY All generators satisfy the given constraint, but there exists a generator not saturating it (i.e., a generator does not belong to the hyper-plane defined by the constraint.).

ALL_SATURATE All generators saturate the given constraint (i.e., they all belong to the hyper-plane defined by the constraint.).

SOME_SATISFY Some generators satisfy the given constraint (i.e., there exists both a generator satisfying the constraint and another generator which does not satisfy it.).

7 PPL Class Documentation

7.1 Parma_Polyhedra_Library::Constraint Class Reference

A linear equality or inequality.

`#include <ppl.hh>`

Inherits Row.

Public Methods

- **Constraint** ()

  Default constructor.

- **Constraint** (const Constraint &c)

  Ordinary copy-constructor.

- ∼**Constraint** ()

  Destructor.

- bool **is_equality** () const

  Returns `true` *if and only if* ∗`this` *is an equality constraint.*

- bool **is_inequality** () const

  Returns `true` *if and only if* ∗`this` *is an inequality constraint.*

Friends

- Constraint **Parma_Polyhedra_Library::operator==** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the constraint `e1 = e2`.

- Constraint **Parma_Polyhedra_Library::operator==** (const **LinExpression** &e, const Integer &n)

  Returns the constraint `e = n`.

- Constraint **Parma_Polyhedra_Library::operator==** (const Integer &n, const **LinExpression** &e)

  Returns the constraint `n = e`.

- Constraint **Parma_Polyhedra_Library::operator>=** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the constraint `e1 >= e2`.

- Constraint **Parma_Polyhedra_Library::operator>=** (const **LinExpression** &e, const Integer &n)

  Returns the constraint `e` >= `n`.

- Constraint **Parma_Polyhedra_Library::operator>=** (const Integer &n, const **LinExpression** &e)

  Returns the constraint `n` >= `e`.

- Constraint **Parma_Polyhedra_Library::operator<=** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the constraint `e1` <= `e2`.

- Constraint **Parma_Polyhedra_Library::operator<=** (const **LinExpression** &e, const Integer &n)

  Returns the constraint `e` <= `n`.

- Constraint **Parma_Polyhedra_Library::operator<=** (const Integer &n, const **LinExpression** &e)

  Returns the constraint `n` <= `e`.

- Constraint **Parma_Polyhedra_Library::operator>>** (const Constraint &c, unsigned int offset)

  Returns the constraint `c` *with variables renamed by adding* `offset` *to their Cartesian axis identifier.*

7.1.1 Detailed Description

An object of the class **Constraint** (p. 9) is either:

- an equality: $\sum_{i=0}^{n-1} a_i x_i + b = 0$; or
- an inequality: $\sum_{i=0}^{n-1} a_i x_i + b \geq 0$;

where n is the dimension of the space.

How to build a constraint

Constraints are typically built by applying a relational operator to a pair of linear expressions. Available relational operators include equality (==) and non-strict inequalities (>= and <=). Strict inequalities (< and >) are not supported.

In the following example it is assumed that variables `x`, `y` and `z` are defined as follows:

```
Variable x(0);
Variable y(1);
Variable z(2);
```

Example

The following code builds the equality $3x + 5y - z = 0$:

```
Constraint equal(3*x + 5*y - z == 0);
```

The following code builds the constraint $4x - 2y \geq z - 13$:

```
Constraint inequal(4*x - 2*y >= z - 13);
```

7.2 Parma_Polyhedra_Library::ConSys Class Reference

A system of constraints.

`#include <ppl.hh>`

Inherits Matrix.

Public Methods

- **ConSys** ()

  Default constructor: builds an empty system of constraints.

- **ConSys** (const ConSys &cs)

  Ordinary copy-constructor.

- virtual ~**ConSys** ()

  Destructor.

- void **insert** (const **Constraint** &c)

  Inserts a copy of the constraint `c` *into* `*this`*, increasing the number of dimensions if needed.*

- void **swap** (ConSys &y)

  Swaps `*this` *with the system of constraints* `y`.

- **const_iterator begin** () const

  Returns the **const_iterator** (p. 12) *pointing to the first constraint, if* `*this` *is not empty; otherwise, returns the past-the-end* **const_iterator** (p. 12).

- **const_iterator end** () const

  Returns the past-the-end **const_iterator** (p. 12).

7.2.1 Detailed Description

An object of the class **ConSys** (p. 10) is a system of constraints, i.e. a multiset of objects of the class **Constraint** (p. 9). When inserting constraints in a system, dimensions are automatically adjusted so that all the constraints in the system are defined on the same vector space.

In all the examples it is assumed that variables `x` and `y` are defined as follows:

```
Variable x(0);
Variable y(1);
```

Example 1

The following code builds a system of constraints corresponding to a square in $\mathbb{R}^2$:

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x <= 3);
cs.insert(y >= 0);
cs.insert(y <= 3);
```

Example 2

The following code builds a system of constraints corresponding to a half-strip in $\mathbb{R}^2$:

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x - y <= 0);
cs.insert(x - y + 1 >= 0);
```

7.3 Parma_Polyhedra_Library::ConSys::const_iterator Class Reference

`#include <ppl.hh>`

Inherits std::iterator.

Public Methods

- **const_iterator** ()

  Default constructor.

- **const_iterator** (const const_iterator &y)

  Ordinary copy-constructor.

- virtual ~**const_iterator** ()

  Destructor.

- const_iterator & **operator=** (const const_iterator &y)

  Assignment operator.

- const **Constraint** & **operator** ∗ () const

  Dereference operator.

- const **Constraint** ∗ **operator** → () const

  Indirect member selector.

- const_iterator & **operator++** ()

  Prefix increment operator.

- const_iterator **operator++** (int)

  Postfix increment operator.

- bool **operator==** (const const_iterator &y) const

  Returns `true` *if and only if* ∗`this` *and* `y` *are identical.*

- bool **operator!=** (const const_iterator &y) const

  Returns `true` *if and only if* ∗`this` *and* `y` *are different.*

7.3.1 Detailed Description

A **const_iterator** (p. 12) is used to provide read-only access to each constraint contained in an object of **ConSys** (p. 10).

Example

The following code prints the system of constraints defining the polyhedron `ph`:

```
const ConSys cs = ph.constraints();
ConSys::const_iterator iend = cs.end();
for (ConSys::const_iterator i = cs.begin(); i != iend; ++i)
  cout << *i << endl;
```

7.4 Parma_Polyhedra_Library::Generator Class Reference

A line, ray or vertex.

`#include <ppl.hh>`

Inherits Row.

Public Types

- enum **Type**

  The generator type.

Public Methods

- **Generator** ()

  Default constructor.

- **Generator** (const Generator &g)

  Ordinary copy-constructor.

- ∼**Generator** ()

  Destructor.

- **Type type** () const

  Returns the generator type of ∗`this`.

Friends

- Generator **Parma_Polyhedra_Library::line** (const **LinExpression** &e)

  Returns the (bidirectional) line of direction `e`.

- Generator **Parma_Polyhedra_Library::ray** (const **LinExpression** &e)

  Returns the (unidirectional) ray of direction `e`.

- Generator **Parma_Polyhedra_Library::vertex** (const **LinExpression** &e, const Integer &d=1)

  Returns the vertex at `e` */* `d` *(note that* `d` *is an optional argument with default value 1).*

  Exceptions:

  std::invalid_argument *thrown if* `d` *is zero.*

7.4.1 Detailed Description

An object of the class **Generator** (p. 13) is one of the following:

- a line $\mathbf{l} = (a_0, \ldots, a_{n-1})^{\mathrm{T}}$;

- a ray $\mathbf{r} = (a_0, \ldots, a_{n-1})^{\mathrm{T}}$;

- a vertex $\mathbf{v} = (\frac{a_0}{d}, \ldots, \frac{a_{n-1}}{d})^{\mathrm{T}}$;

where n is the dimension of the space.

How to build a generator.

Each type of generator is built by applying the corresponding function (`line`, `ray` or `vertex`) to a linear expression, representing a direction in the space. This means that a linear expression used to define a generator should be homogeneous and any constant term will be ignored. When defining a vertex, an optional Integer argument can be used as a common *denominator* for all the coefficients occurring in the provided linear expression; the default value for this argument is 1.

In all the following examples it is assumed that variables `x`, `y` and `z` are defined as follows:

```
Variable x(0);
Variable y(1);
Variable z(2);
```

Example 1

The following code builds a line with direction $x - y - z$:

```
Generator l = line(x - y - z);
```

As mentioned above, the constant term of the linear expression is not relevant. Thus, the following code has the same effect:

```
Generator l = line(x - y - z + 15);
```

Example 2

The following code builds a ray with the same direction as the line in Example 1:

```
Generator r = ray(x - y - z);
```

As is the case for lines, when specifying a ray the constant term of the linear expression is not relevant.

Example 3

The following code builds the vertex $\mathbf{v} = (1, 0, 2)^{\mathrm{T}} \in \mathbb{R}^3$:

```
Generator v = vertex(1*x + 0*y + 2*z);
```

The same effect can be obtained by using the following code:

```
Generator v = vertex(x + 2*z);
```

Similarly, the origin $\mathbf{0} \in \mathbb{R}^3$ can be defined using either one of the following lines of code:

```
Generator origin1 = vertex(0*x + 0*y + 0*z);
Generator origin2 = vertex(0*z);
```

Note however that the following line would have defined a different vertex, namely $\mathbf{0} \in \mathbb{R}^2$:

```
Generator origin3 = vertex(0*y);
```

Example 4

The vertex $\mathbf{v}$ specified in Example 3 above can also be obtained with the following code, where we provide a non-default value for the denominator argument:

```
Generator v = vertex(2*x + 0*y + 4*z, 2);
```

Obviously, the denominator can be usefully exploited to specify vertices having some non-integer (but rational) coordinates. For instance, the vertex $\mathbf{w} = (-1.5, 3.2, 2.1)^{\mathrm{T}} \in \mathbb{R}^3$ can be specified by the following code:

```
Generator w = vertex(-15*x + 32*y + 21*z, 10);
```

If a zero denominator is provided, an exception is thrown.

7.5 Parma_Polyhedra_Library::GenSys Class Reference

A system of generators.

`#include <ppl.hh>`

Inherits Matrix.

Public Methods

- **GenSys** ()

  Default constructor: builds an empty system of generators.

- **GenSys** (const GenSys &gs)

  Ordinary copy-constructor.

- virtual ~**GenSys** ()

  Destructor.

- void **insert** (const **Generator** &g)

  Inserts a copy of the generator `g` *into* `*this`*, increasing the number of dimensions if needed.*

- void **swap** (GenSys &y)

  Swaps `*this` *with the system of generators* `y`.

- **const_iterator begin** () const

  Returns the **const_iterator** (p. 16) *pointing to the first generator, if* `*this` *is not empty; otherwise, returns the past-the-end* **const_iterator** (p. 16).

- **const_iterator end** () const

  Returns the past-the-end **const_iterator** (p. 16).

7.5.1 Detailed Description

An object of the class **GenSys** (p. 15) is a system of generators, i.e. a multiset of objects of the class **Generator** (p. 13) (lines, rays and vertices). When inserting generators in a system, dimensions are automatically adjusted so that all the generators in the system are defined on the same vector space. A system of generators which is meant to define a non-empty polyhedron must include at least one vertex, even if the polyhedron has no "proper" vertices: the reason is that lines and rays need a supporting point (they only specify directions).

In all the examples it is assumed that variables `x` and `y` are defined as follows:

```
Variable x(0);
Variable y(1);
```

Example 1

The following code defines the line having the same direction as the x axis (i.e., the first Cartesian axis) in $\mathbb{R}^2$:

```
GenSys gs;
gs.insert(line(x + 0*y));
```

As said above, this system of generators corresponds to an empty polyhedron, because the line has no supporting point. To define a system of generators indeed corresponding to the x axis, one can add the following code which inserts the origin of the space as a vertex:

```
gs.insert(vertex(0*x + 0*y));
```

Since dimensions are automatically adjusted, the following code obtains the same effect:

```
gs.insert(vertex(0*x));
```

In contrast, if we had added the following code, we would have defined a line parallel to the x axis and including the point $(0, 1)^{\mathrm{T}} \in \mathbb{R}^2$.

```
gs.insert(vertex(0*x + 1*y));
```

Example 2

The following code builds a ray having the same direction as the positive part of the x axis in $\mathbb{R}^2$:

```
GenSys gs;
gs.insert(ray(x + 0*y));
```

To define a system of generators indeed corresponding to the set

$$\{ (x, 0)^{\mathrm{T}} \in \mathbb{R}^2 \mid x \geq 0 \},$$

one just has to add the origin:

```
gs.insert(vertex(0*x + 0*y));
```

Example 3

The following code builds a system of generators having four vertices and corresponding to a square in $\mathbb{R}^2$ (the same as Example 1 for the system of constraints):

```
GenSys gs;
gs.insert(vertex(0*x + 0*y));
gs.insert(vertex(0*x + 3*y));
gs.insert(vertex(3*x + 0*y));
gs.insert(vertex(3*x + 3*y));
```

Example 4

The following code builds a system of generators having two vertices and a ray, corresponding to a half-strip in $\mathbb{R}^2$ (the same as Example 2 for the system of constraints):

```
GenSys gs;
gs.insert(vertex(0*x + 0*y));
gs.insert(vertex(0*x + 1*y));
gs.insert(ray(x - y));
```

7.6 Parma_Polyhedra_Library::GenSys::const_iterator Class Reference

```
#include <ppl.hh>
```

Inherits std::iterator.

Public Methods

- **const_iterator** ()
  Default constructor.
- **const_iterator** (const const_iterator &y)
  Ordinary copy-constructor.
- virtual ∼**const_iterator** ()
  Destructor.
- const_iterator & **operator=** (const const_iterator &y)
  Assignment operator.
- const **Generator** & **operator** ∗ () const
  Dereference operator.
- const **Generator** ∗ **operator** → () const
  Indirect member selector.
- const_iterator & **operator++** ()
  Prefix increment operator.
- const_iterator **operator++** (int)
  Postfix increment operator.
- bool **operator==** (const const_iterator &y) const
  Returns `true` if and only if ∗`this` and `y` are identical.
- bool **operator!=** (const const_iterator &y) const
  Returns `true` if and only if ∗`this` and `y` are different.

7.6.1 Detailed Description

A **const_iterator** (p. 16) is used to provide read-only access to each generator contained in an object of **GenSys** (p. 15).

Example

The following code prints the system of generators of the polyhedron `ph`:

```
const GenSys gs = ph.generators();
GenSys::const_iterator iend = gs.end();
for (GenSys::const_iterator i = gs.begin(); i != iend; ++i)
  cout << *i << endl;
```

7.7 Parma_Polyhedra_Library::LinExpression Class Reference

A linear expression.

`#include <ppl.hh>`

Inherits Row.

Public Methods

- **LinExpression** ()

  Default constructor.

- **LinExpression** (const LinExpression &e)

  Ordinary copy-constructor.

- virtual ∼**LinExpression** ()

  Destructor.

- **LinExpression** (const Integer &n)

  Constructor: builds the linear expression corresponding to the inhomogeneous term `n`.

- **LinExpression** (const **Variable** &v)

  Constructor: builds the linear expression corresponding to the variable `v`.

Friends

- LinExpression **Parma_Polyhedra_Library::operator+** (const LinExpression &e1, const LinExpression &e2)

  Returns the linear expression `e1 + e2`.

- LinExpression **Parma_Polyhedra_Library::operator+** (const Integer &n, const LinExpression &e)

  Returns the linear expression `n + e`.

- LinExpression **Parma_Polyhedra_Library::operator+** (const LinExpression &e, const Integer &n)

  Returns the linear expression `e + n`.

- LinExpression **Parma_Polyhedra_Library::operator-** (const LinExpression &e)

  Returns the linear expression `- e`.

- LinExpression **Parma_Polyhedra_Library::operator-** (const LinExpression &e1, const LinExpression &e2)

  Returns the linear expression `e1 - e2`.

- LinExpression **Parma_Polyhedra_Library::operator-** (const Integer &n, const LinExpression &e)

  Returns the linear expression `n - e`.

- LinExpression **Parma_Polyhedra_Library::operator-** (const LinExpression &e, const Integer &n)

  Returns the linear expression `e - n`.

- LinExpression **Parma_Polyhedra_Library::operator** ∗ (const Integer &n, const LinExpression &e)

  Returns the linear expression `n * e`.

- LinExpression **Parma_Polyhedra_Library::operator** ∗ (const LinExpression &e, const Integer &n)

  Returns the linear expression `e * n`.

- LinExpression & **Parma_Polyhedra_Library::operator+=** (LinExpression &e1, const LinExpression &e2)

  Returns the linear expression `e1 + e2` *and assigns it to* `e1`.

- LinExpression & **Parma_Polyhedra_Library::operator+=** (LinExpression &e, const **Variable** &v)

  Returns the linear expression `e + v` *and assigns it to* `e`.

- LinExpression & **Parma_Polyhedra_Library::operator+=** (LinExpression &e, const Integer &n)

  Returns the linear expression `e + n` *and assigns it to* `e`.

7.7.1 Detailed Description

An object of the class **LinExpression** (p. 17) represents the linear expression

$$\sum_{i=0}^{n-1} a_i x_i + b$$

where n is the dimension of the space, each a_i is the integer coefficient of the `i` -th variable x_i and b is the integer for the inhomogeneous term.

How to build a linear expression.

Linear expressions are the basic blocks for defining both constraints (i.e., linear equalities or inequalities) and generators (i.e., lines, rays and vertices). The following functions provide a convenient interface for building a complex linear expression starting from simpler ones (or even from objects of the classes **Variable** (p. 26) and Integer). Available operators include unary negation, binary addition and subtraction, as well as multiplication by an Integer.

Example

The following code builds the linear expression $4x - 2y - z + 14$:

```
LinExpression e = 4*x - 2*y - z + 14;
```

Another way to build the same linear expression is:

```
LinExpression e1 = 4*x;
LinExpression e2 = 2*y;
LinExpression e3 = z;
LinExpression e = LinExpression(14);
e += e1 - e2 - e3;
```

7.8 Parma_Polyhedra_Library::Polyhedron Class Reference

A convex polyhedron.

`#include <ppl.hh>`

Public Types

- enum **Degenerate_Kind** { **ZERO_DIMENSIONAL**, **EMPTY** }

  Kinds of degenerate polyhedra.

Public Methods

- **Polyhedron** (**Degenerate_Kind** kind=ZERO_DIMENSIONAL)

  Builds the zero-dimensional, universe polyhedron, if `kind` *is* `ZERO_DIMENSIONAL` *(the default); otherwise (i.e., if* `kind` *is* `EMPTY`*) builds an empty polyhedron.*

- **Polyhedron** (const Polyhedron &y)

  Ordinary copy-constructor.

- **Polyhedron** (size_t num_dimensions)

  Builds the universe polyhedron of dimension `num_dimensions`.

- **Polyhedron** (**ConSys** &cs)

  Builds a polyhedron from a system of constraints.

  Parameters:

  cs *The system of constraints defining the polyhedron. It is not declared* `const` *because it can be modified.*

- **Polyhedron** (**GenSys** &gs)

  Builds a polyhedron from a system of generators.

  Parameters:

  gs *The system of generators defining the polyhedron. It is not declared* `const` *because it can be modified.*

  Exceptions:

  std::invalid_argument *thrown if the system of generators has no vertex.*

- Polyhedron & **operator=** (const Polyhedron &y)

  The assignment operator.

- size_t **num_dimensions** () const

  Returns the dimension of the polyhedron.

- void **intersection_assign** (const Polyhedron &y)

  Intersects ∗`this` *with polyhedron* `y` *and assigns the result to* ∗`this`.

  Exceptions:

  std::invalid_argument *thrown if* ∗`this` *and* `y` *have different dimension.*

- void **convex_hull_assign** (const Polyhedron &y)

  Assigns the convex hull of ∗`this` ∪ `y` *to* ∗`this`.

  Exceptions:

  std::invalid_argument *thrown if* ∗`this` *and* `y` *have different dimension.*

- void **convex_hull_assign_lazy** (const Polyhedron &y)

  Assigns the convex hull of ∗`this` ∪ `y` *to* ∗`this`*, without minimizing the result.*

  Exceptions:

  std::invalid_argument *thrown if* ∗`this` *and* `y` *have different dimension.*

- **GenSys_Con_Rel satisfies** (const **Constraint** &c)

  Returns the relation between the generators of ∗`this` *and the constraint* `c`.

  Exceptions:

  std::invalid_argument *thrown if*∗`this` *and constraint* `c` *have different dimension.*

- bool **includes** (const **Generator** &g)

  Tests the inclusion of the generator `g` *in the polyhedron* `*this`.

  Exceptions:

  std::invalid_argument *thrown if* `*this` *and constraint* `g` *have different dimension.*

- void **widening_assign** (const Polyhedron &y)

  Computes the widening between `*this` *and* `y` *and assigns the result to* `*this`.

  Parameters:

  y *The polyhedron that* must *be contained in* `*this`.

  Exceptions:

  std::invalid_argument *thrown if* `*this` *and* `y` *have different dimension.*

- bool **limited_widening_assign** (const Polyhedron &y, **ConSys** &cs)

  Limits the widening between `*this` *and* `y` *by* `cs` *and assigns the result to* `*this`.

  Parameters:

  y *The polyhedron that* must *be contained in* `*this`.

  cs *The system of constraints that limits the widened polyhedron. It is not declared* `const` *because it can be modified.*

  Returns:

  `true` *if the resulting polyhedron is not empty* `false` *otherwise.*

  Exceptions:

  std::invalid_argument *thrown if* `*this`, `y` *and* `cs` *have different dimension.*

- const **ConSys** & **constraints** () const

  Returns the system of constraints.

  Exceptions:

  std::invalid_argument *thrown if* `*this` *is empty.*

- const **GenSys** & **generators** () const

  Returns the system of generators.

  Exceptions:

  std::invalid_argument *thrown if* `*this` *is zero-dimensional.*

- void **insert** (const **Constraint** &c)

  Inserts a new constraint `c` *into the system of constraints of* `*this`.

- void **insert** (const **Generator** &g)

  Inserts a new generator `g` *into the system of generators of* `*this`.

- void **assign_variable** (const **Variable** &v, const **LinExpression** &expr, const Integer &denominator=1)

  Assigns an affine expression to the specified variable.

  Parameters:

  v *The variable to which the affine expression is assigned.*

  expr *The numerator of the affine expression.*

  denominator *The denominator of the affine expression (optional argument with default value 1.)*

  Exceptions:

  std::invalid_argument *thrown if* `denominator` *is zero or if* `expr` *and* `*this` *have different dimension or if* `v` *is not a variable of the polyhedron.*

- void **substitute_variable** (const **Variable** &v, const **LinExpression** &expr, const Integer &denominator=1)

  Substitutes an affine expression for the specified variable.

Parameters:

v *The variable to which the affine expression is substituted.*
expr *The numerator of the affine expression.*
denominator *The denominator of the affine expression (optional argument with default value 1.)*

Exceptions:

std::invalid_argument *thrown if* `denominator` *is zero or if* `expr` *and* `*this` *have different dimension or if* `v` *is not a variable of the polyhedron.*

- bool **OK** (bool check_not_empty=true) const

  Checks if all the invariants are satisfied.

  Parameters:

  check_not_empty `true` *if it must be checked whether the system of constraint is satisfiable.*

  Returns:

  `true` *if the polyhedron satisfies all the invariants stated in the PPL,* `false` *otherwise.*

- void **add_dimensions_and_embed** (size_t dim)

  Adds new dimensions and embeds the old polyhedron in the new space.

  Parameters:

  dim *The number of dimensions to add.*

- void **add_dimensions_and_project** (size_t dim)

  Adds new dimensions to the polyhedron and does not embed it in the new space.

  Parameters:

  dim *The number of dimensions to add.*

- void **remove_dimensions** (const std::set< **Variable** > &to_be_removed)

  Removes the specified dimensions.

  Parameters:

  to_be_removed *The set of variables to remove.*

- bool **add_constraints** (**ConSys** &cs)

  Adds the specified constraints and computes a new polyhedron.

  Parameters:

  cs *The constraints that will be added to the current system of constraints. This parameter is not declared* `const` *because it can be modified.*

  Returns:

  `false` *if the resulting polyhedron is empty.*

  Exceptions:

  std::invalid_argument *thrown if* `*this` *and* `cs` *have different dimension.*

- void **add_constraints_lazy** (**ConSys** &cs)

  Adds the specified constraints without minimizing.

  Parameters:

  cs *The constraints that will be added to the current system of constraints. This parameter is not declared* `const` *because it can be modified.*

  Exceptions:

  std::invalid_argument *thrown if* `*this` *and* `cs` *have different dimension.*

- void **add_generators** (**GenSys** &gs)

  Adds the specified generators.

  Parameters:

  gs *The generators that will be added to the current system of generators. The parameter is not declared* `const` *because it can be modified.*

  Exceptions:

  std::invalid_argument *thrown if* `*this` *and* `gs` *have different dimension.*

- bool **check_empty** () const
  Returns `true` *if and only if the polyhedron is empty.*

- bool **check_universe** () const
  Returns `true` *if* ∗`this` *is a universe polyhedron.*

- void **swap** (Polyhedron &y)
  Swaps ∗`this` *with polyhedron* `y`.

- bool **is_empty** () const
  Returns `true` *if and only if* ∗`this` *is an empty polyhedron.*

- bool **is_zero_dim** () const
  Returns `true` *if and only if* ∗`this` *is a zero-dimensional polyhedron.*

Friends

- bool **Parma_Polyhedra_Library::operator**<= (const Polyhedron &x, const Polyhedron &y)
  Returns `true` *if and only if polyhedron* `x` *is contained in polyhedron* `y`.

- std::ostream & **Parma_Polyhedra_Library::operator**<< (std::ostream &s, const Polyhedron &p)
  Output operator.

- std::istream & **Parma_Polyhedra_Library::operator**>> (std::istream &s, Polyhedron &p)
  Input operator.

7.8.1 Detailed Description

An object of the class **Polyhedron** (p. 19) represents a convex polyhedron in the space $\mathbb{R}^n$.

A polyhedron can be specified as either a finite system of constraints or a finite system of generators (see Minkowski's theorem in the Introduction). So, it is possible to obtain one system from the other. That is, if we know the system of constraints, we can obtain from this the system of generators that define the same polyhedron and vice versa. These systems can contain some redundant members: in this case we say that they are not in the minimal form.

In all the examples it is assumed that variables `x` and `y` are defined (where they are used) as follows:

```
Variable x(0);
Variable y(1);
```

Example 1

The following code builds a polyhedron corresponding to a square in $\mathbb{R}^2$, given as a system of constraints:

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x <= 3);
cs.insert(y >= 0);
cs.insert(y <= 3);
Polyhedron ph(cs);
```

The following code builds the same polyhedron as above, but starting from a system of generators specifying the four vertices of the square:

```
GenSys gs;
gs.insert(vertex(0*x + 0*y));
gs.insert(vertex(0*x + 3*y));
gs.insert(vertex(3*x + 0*y));
gs.insert(vertex(3*x + 3*y));
Polyhedron ph(gs);
```

Example 2

The following code builds an unbounded polyhedron corresponding to a half-strip in $\mathbb{R}^2$, given as a system of constraints:

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x - y <= 0);
cs.insert(x - y + 1 >= 0);
Polyhedron ph(cs);
```

The following code builds the same polyhedron as above, but starting from the system of generators specifying the two vertices of the polyhedron and one ray:

```
GenSys gs;
gs.insert(vertex(0*x + 0*y));
gs.insert(vertex(0*x + y));
gs.insert(ray(x - y));
Polyhedron ph(gs);
```

Example 3

The following code builds the polyhedron corresponding to an half-plane in $\mathbb{R}^2$, by adding a single constraint to the universe polyhedron:

```
Polyhedron ph;
ph.insert(y >= 0);
```

The following code builds the same polyhedron as above, but starting from a system of generators specifying a vertex, a ray and a line.

```
Polyhedron ph;
ph.insert(vertex(0*x + 0*y));
ph.insert(ray(0*x + y));
ph.insert(line(x + 0*y));
```

In this last case, it is important to note that: even if this polyhedron has no real vertex, we must add one, because otherwise the polyhedron is considered empty.

Example 4

The following code shows the use of the function `add_dimensions_and_embed`:

```
Polyhedron ph;
ph.insert(x == 2);
ph.add_dimensions_and_embed(1);
```

We start with the universe polyhedron in the 0-dimensional space. Then we add a single equality constraint, thus obtaining the polyhedron corresponding to the singleton set $\{2\} \subseteq \mathbb{R}$. After the last line of code, the resulting polyhedron is

$$\left\{ (2, x_1)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; x_1 \in \mathbb{R} \right\}.$$

Example 5

The following code shows the use of the function `add_dimensions_and_project`:

```
Polyhedron ph;
ph.insert(x == 2);
ph.add_dimensions_and_poject(1);
```

The first two lines of code are the same as in Example 4 for `add_dimensions_and_embed`. After the last line of code, the resulting polyhedron is the singleton set $\{(2,0)^{\mathrm{T}}\} \subseteq \mathbb{R}^2$.

Example 6

The following code shows the use of the function `assign_variable`:

```
Polyhedron ph;
ph.insert(vertex(0*x + 0*y));
ph.insert(vertex(0*x + 3*y));
ph.insert(vertex(3*x + 0*y));
ph.insert(vertex(3*x + 3*y));
LinExpression coeff = x + 0*y + 4;
ph.assign_variable(x, coeff);
```

In this example the starting polyhedron is a square in $\mathbb{R}^2$, the considered variable is x and the affine expression is $x + 4$. The resulting polyhedron is the same square translated towards right. Moreover, if the affine transformation for the same variable `x` is $x + y$:

```
LinExpression coeff = x + y;
```

the resulting polyhedron is a parallelogram with the height equal to the side of the square and the oblique sides parallel to the line $x - y$. Instead, if we do not use an invertible transformation for the same variable; for example, the affine expression y:

```
LinExpression coeff = 0*x + y;
```

the resulting polyhedron is a diagonal of the square.

Example 7

The following code shows the use of the function `substitute_variable`:

```
Polyhedron ph;
ph.insert(x >= 0);
ph.insert(x <= 3);
ph.insert(y >= 0);
ph.insert(y <= 3);
LinExpression coeff = x + 0*y + 4;
ph.substitute_variable(x, coeff);
```

In this example the starting polyhedron, `var` and the affine expression and the denominator are the same as in Example 6, while the resulting polyhedron is again the same square but translated towards left. Moreover, if the affine transformation for `x` is $x + y$

```
LinExpression coeff = x + y;
```

the resulting polyhedron is a parallelogram with the height equal to the side of the square and the oblique sides parallel to the line $x + y$. Instead, if we do not use an invertible transformation for the same variable `x`, for example, the affine expression y:

```
LinExpression coeff = 0*x + y;
```

the resulting polyhedron is a line that corresponds to the y axis.

7.8.2 Member Enumeration Documentation

7.8.2.1 enum Parma_Polyhedra_Library::Polyhedron::Degenerate_Kind

Enumeration values:

ZERO_DIMENSIONAL The full polyhedron in $\mathbb{R}^0$, i.e., a singleton.

EMPTY The empty polyhedron, i.e., the empty set.

7.9 Parma_Polyhedra_Library::Variable Class Reference

A dimension of the space.

`#include <ppl.hh>`

Public Methods

- **Variable** (unsigned int id)

  Constructor: `id` *is the index of the Cartesian axis.*

- unsigned int **id** () const

  Returns the index of the Cartesian axis.

7.9.1 Detailed Description

An object of the class **Variable** (p. 26) represents a dimension of the space, that is one of the Cartesian axes. Variables are used as base blocks in order to build more complex linear expressions. Each variable is identified by a non-negative integer, representing the index of the corresponding Cartesian axis (the first axis has index 0).

Note that the “meaning” of an object of the class **Variable** (p. 26) is completely specified by the integer index provided to its constructor: be careful not to be mislead by C++ language variable names. For instance, in the following example the linear expressions `e1` and `e2` are equivalent, since the two variables `x` and `z` denote the same Cartesian axis.

```
Variable x(0);
Variable y(1);
Variable z(0);
LinExpression e1 = x + y;
LinExpression e2 = y + z;
```

8 PPL Page Documentation

8.1 GNU GENERAL PUBLIC LICENSE

Version 2, June 1991

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