
The Parma Polyhedra Library

User's Manual*

(version 0.1)

Roberto Bagnara[†]

Sara Bonini

Patricia M. Hill[‡]

Andrea Pescetti

Elisa Ricci[§]

Angela Stazzone

Enea Zaffanella[¶]

Tatiana Zolo^{||}

October 23, 2001

*This work has been partly supported by: University of Parma's FIL scientific research project (ex 60%) "Pure and Applied Mathematics"; MURST project "Automatic Program Certification by Abstract Interpretation"; MURST project "Abstract Interpretation, Type Systems and Control-Flow Analysis".

[†]bagnara@cs.unipr.it, Department of Mathematics, University of Parma, Italy.

[‡]hill@comp.leeds.ac.uk, School of Computing, University of Leeds, U.K.

[§]ericci@cs.unipr.it, Department of Mathematics, University of Parma, Italy.

[¶]zaffanella@cs.unipr.it, Department of Mathematics, University of Parma, Italy.

^{||}zolo@cs.unipr.it, Department of Mathematics, University of Parma, Italy.

Copyright © 2001 Roberto Bagnara (bagnara@cs.unipr.it).

This document describes the Parma Polyhedra Library (PPL).

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.1 or any later version published by the **Free Software Foundation**; with no Invariant Sections, with no Front-Cover Texts, and with no Back-Cover Texts. A copy of the license is included in the section entitled “**GNU Free Documentation License**”.

The PPL is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the **Free Software Foundation**; either version 2 of the License, or (at your option) any later version. A copy of the license is included in the section entitled “**GNU GENERAL PUBLIC LICENSE**”.

The PPL is distributed in the hope that it will be useful, but WITHOUT ANY WARRANTY; without even the implied warranty of MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the GNU General Public License for more details.

For the most up-to-date information see the Parma Polyhedra Library WWW site:

<http://www.cs.unipr.it/ppl/>

Contents

1	Convex Polyhedra and the PPL	1
2	PPL Namespace Index	5
3	PPL Hierarchical Index	6
4	PPL Compound Index	6
5	PPL Page Index	7
6	PPL Namespace Documentation	7
7	PPL Class Documentation	9
8	PPL Page Documentation	26

1 Convex Polyhedra and the PPL

1.1 An Introduction to Convex Polyhedra

The following definitions and results are taken from:

- G. L. Nemhauser and L. A. Wolsey - Integer and Combinatorial Optimization - Wiley Interscience Series in Discrete Mathematics and Optimization, 1988.
- D. K. Wilde - A library for doing polyhedral operations - IRISA Publication interne n. 785, December 1993.

- K. Fukuda - Polyhedral Computation FAQ - Swiss Federal Institute of Technology, Lausanne and Zurich, Switzerland, October 2000.

Combination

Let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$. The linear combination $\sum_{j=1}^k \lambda_j \mathbf{x}_j$ is said to be

- a *positive combination*, if $\forall j \in \{1, \dots, k\} : \lambda_j \geq 0$;
- an *affine combination*, if $\sum_{j=1}^k \lambda_j = 1$;
- a *convex combination*, if both the previous conditions hold.

Note that when $k = 0$, $\sum_{j=1}^k \lambda_j \mathbf{x}_j = \mathbf{0}$ and $\sum_{j=1}^k \lambda_j = 0$. This means that $\sum_{j=1}^0 \lambda_j \mathbf{x}_j$ may be regarded as a positive but not an affine combination.

Scalar product

Let $\mathbf{x} = (x_0, \dots, x_{n-1})^T, \mathbf{y} = (y_0, \dots, y_{n-1})^T \in \mathbb{R}^n$. The *scalar product* of \mathbf{x} and \mathbf{y} is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=0}^{n-1} x_i y_i.$$

The vectors \mathbf{x} and \mathbf{y} are *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Convex hull

The *convex hull* of a set $K \subseteq \mathbb{R}^n$ is the set of all the convex combinations of the points in K . The set K is convex if it is its own convex hull.

Affine transformation

An *affine transformation* is a function mapping a point $\mathbf{x} \in \mathbb{R}^n$ to a point $\mathbf{x}' \in \mathbb{R}^m$ such that

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}$$

where $A \in \mathbb{R}^m \times \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$.

Linear independence

A set of points $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$ is *linearly independent* if, for all $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, the set of equations

$$\sum_{i=1}^k \lambda_i \mathbf{x}_i = \mathbf{0}$$

implies that, for each $i = 1, \dots, k$, $\lambda_i = 0$.

Note that the maximum number of linearly independent points in \mathbb{R}^n is n .

Proposition

If A is an $m \times n$ matrix, the maximum number of linearly independent rows of A , viewed as vectors of \mathbb{R}^n , equals the maximum number of linearly independent columns of A , viewed as vectors of \mathbb{R}^m .

Rank

The maximum number of linearly independent rows (columns) of a matrix A is the *rank* of A and is denoted by $\text{rank}(A)$.

Affine independence

A set of points $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$ is *affinely independent* if, for all $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, the set of equations

$$\sum_{i=1}^k \lambda_i \mathbf{x}_i = \mathbf{0}, \quad \sum_{i=1}^k \lambda_i = 0$$

implies that, for each $i = 1, \dots, k$, $\lambda_i = 0$.

Note that linear independence implies affine independence, but the converse is not true. Moreover the maximum number of affinely independent points in \mathbb{R}^n is $n + 1$ (e.g., n linearly independent points and the origin $\mathbf{0}$).

Polyhedron

A set $P \subseteq \mathbb{R}^n$ is called a *polyhedron* if it is the set of solutions to a finite number of linear equalities and inequalities:

$$P = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}, C\mathbf{x} \geq \mathbf{d} \},$$

where, if m_1 is the number of linear equalities and m_2 the number of linear inequalities, $A \in \mathbb{R}^{m_1} \times \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^{m_1}$, $C \in \mathbb{R}^{m_2} \times \mathbb{R}^n$ and $\mathbf{d} \in \mathbb{R}^{m_2}$.

In the sequel, we will simply write “equality” and “inequality” to mean “linear equality” and “linear inequality”, respectively; also, we will refer to either an equality or an inequality as a *constraint*.

Constraints representation

It follows that a polyhedron $P \subseteq \mathbb{R}^n$ can be always represented by a *system of constraints*:

$$P = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \}$$

for some matrix $A \in \mathbb{R}^m \times \mathbb{R}^n$ and vector $\mathbf{b} \in \mathbb{R}^m$.

Note that, if $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ and the system of constraints contains the two inequalities $\langle \mathbf{c}, \mathbf{x} \rangle \geq \lambda$ and $\langle \mathbf{c}, \mathbf{x} \rangle \leq \lambda$ (i.e., $\langle -\mathbf{c}, \mathbf{x} \rangle \geq -\lambda$), then they can be replaced by the equivalent unique *equality* $\langle \mathbf{c}, \mathbf{x} \rangle = \lambda$. Conversely, if we have an equality, then it can be replaced by two inequalities (as above).

Rational polyhedron

A polyhedron $P \subseteq \mathbb{R}^n$ is said to be *rational* if there exists a matrix $A \in \mathbb{R}^{m'} \times \mathbb{R}^n$ and a vector $\mathbf{b} \in \mathbb{R}^{m'}$ with rational coefficients such that

$$P = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \}.$$

In the sequel, we will consider only rational polyhedra and assume that, if $\{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \}$ is a system of constraints representing a polyhedron, then A and \mathbf{b} have rational coefficients.

Universe polyhedron

A polyhedron $P \subseteq \mathbb{R}^n$ is called *universe polyhedron* if it is the whole space (i.e. $P = \mathbb{R}^n$).

Polytope

A polyhedron $P \subset \mathbb{R}^n$ is *bounded* if there exists a $\lambda \in \mathbb{R}$, $\lambda > 0$ such that

$$P \subseteq \{ (x_0, \dots, x_{n-1})^T \in \mathbb{R}^n \mid -\lambda \leq x_j \leq \lambda \text{ for } j = 0, \dots, n-1 \}.$$

A bounded polyhedron is called a *polytope*.

Proposition

A polyhedron is a closed convex set.

Dimension

A polyhedron $P \subseteq \mathbb{R}^n$ is of *dimension* k , denoted by $\dim(P) = k$, if the maximum number of affinely independent points in P is $k + 1$.

Vertex

A *vertex* of a polyhedron P is any point in P which cannot be expressed as a convex combination of any other distinct points in P .

Ray

Let P, P_0 be the polyhedra

$$P = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \} \neq \emptyset \quad \text{and} \quad P_0 = \{ \mathbf{r} \in \mathbb{R}^n \mid A\mathbf{r} \geq \mathbf{0} \}$$

where $A \in \mathbb{R}^m \times \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$. Then any point $\mathbf{r} \in P_0 \setminus \{\mathbf{0}\}$ is called a *ray* of P .

A ray indicates a direction in which the polyhedron P is infinite (i.e., unbounded).

Proposition

A point $\mathbf{r} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is a ray of a non-empty polyhedron $P \subseteq \mathbb{R}^n$ if and only if, for any point $\mathbf{x} \in P$, $(\mathbf{x} + \mu\mathbf{r}) \in P$ for all $\mu \in \mathbb{R}, \mu > 0$.

Extreme ray

A ray \mathbf{r} of a polyhedron P is an *extreme ray* if there do not exist two rays \mathbf{r}_1 and \mathbf{r}_2 of P , where $\mathbf{r}_1 \neq \lambda\mathbf{r}_2$ for any $\lambda \in \mathbb{R}, \lambda > 0$, such that

$$\mathbf{r} = \mu_1\mathbf{r}_1 + \mu_2\mathbf{r}_2,$$

where $\mu_1, \mu_2 \in \mathbb{R}, \mu_1 > 0$ and $\mu_2 > 0$.

Line

A *line* (or *bidirectional ray*) of a polyhedron $P \subseteq \mathbb{R}^n$ is a ray \mathbf{l} of P such that $-\mathbf{l}$ is another ray of P .

Cone

A set $C \subseteq \mathbb{R}^n$ is a *cone* if

$$\mathbf{x} \in C \Rightarrow \lambda\mathbf{x} \in C \text{ for all } \lambda \in \mathbb{R}, \lambda \geq 0.$$

Polyhedral cone

The polyhedron $P = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{0} \}$ is a convex cone and is called *polyhedral cone*.

Thus, a polyhedral cone is either *pointed*, having the origin as its only vertex, or has no vertices at all.

Lineality space

Given a polyhedron $P = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \}$, the *lineality space* of P is the set

$$\{ \mathbf{x} \in P \mid A\mathbf{x} = \mathbf{0} \}$$

and it is denoted by $\text{lin.space}(P)$.

Minkowski's sum

Let $R, S \subseteq \mathbb{R}^n$ be two sets of vectors. Then the *Minkowski's sum* of R and S is:

$$R + S = \{ r + s \mid r \in R, s \in S \}.$$

Generators representation

A polyhedron $P \subseteq \mathbb{R}^n$ can also be represented by a finite set V of points of P , a finite set R of rays of P and a finite set L of lines of P . The elements of these three sets are the *generators* of P , in the sense that

$$P = \mathcal{V} + \mathcal{R} + \mathcal{L},$$

where the symbol '+' denotes the Minkowski's sum and

- \mathcal{V} is the set of all the convex combinations of the points in V ;
- \mathcal{R} is the set of all the non-negative combinations of the rays in R ; and
- \mathcal{L} is the set of all the linear combinations of the lines in L .

Note that: \mathcal{V} is a polytope, \mathcal{R} is a pointed cone, and \mathcal{L} is $\text{lin.space}(P)$.

Note also that V must contain all vertices of P . However, V can contain other points, particularly if P is a non-empty polyhedron having no vertices (e.g., a half-space).

In the case that P contains at least one vertex, (in which case, $L = \emptyset$) the following two theorems justify this terminology.

Minkowski's theorem

Let $P = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b} \}$ be a non-empty polyhedron where $\text{rank}(A) = n$. Let V be the set of vertices and R the set of extreme rays of P . Let also \mathcal{V} be the set of convex combinations of V and \mathcal{R} the set of positive combinations of R . Then

$$P = \mathcal{V} + \mathcal{R}.$$

The conditions that P is not empty and $\text{rank}(A) = n$ required by this theorem are equivalent to the condition that P has a vertex. This condition is needed since, if the set of vertices $V = \emptyset$, then $\mathcal{V} = \emptyset$ and hence $\mathcal{V} + \mathcal{R}$ is also empty even though P may contain a line. (See also Nemhauser and Wolsey - Integer and Combinatorial Optimization - propositions 4.1 and 4.2 on pages 92 and 93).

The second theorem, called Weil's theorem, states that, starting from a system of generators (having rational coefficients), we can build a rational polyhedron:

Weil's theorem

If A is a rational $m \times n$ matrix, B is a rational $m' \times n$ matrix and

$$Q = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \begin{array}{l} \mathbf{x}^T = \mathbf{y}^T A + \mathbf{z}^T B, \\ \mathbf{y} = (y_0, \dots, y_{m-1})^T \in \mathbb{R}_+^m, \sum_{k=0}^{m-1} y_k = 1, \\ \mathbf{z} \in \mathbb{R}_+^{m'} \end{array} \right\}.$$

then Q is a rational polyhedron.

In fact, since Q consists of the sum of convex combinations of the rows of A with positive combinations of the rows of B , we can think of A as the matrix of vertices and B as the matrix of rays.

Dual representation

Thus a rational polyhedron P has a *dual representation*. That is, P can be represented by a system of constraints or a system of generators. Moreover, given one of the representations, there is an algorithm for computing the other.

(The following spurious string of characters in the user manual is due to a bug in Doxygen.)

\section{\f in\Rset{\f\f vect in\Rset{\f\f vect\ xi\ vect\ xi\ transpose in\Rset{\f\f xi\ geq{\f\f\f\f\f\f\f\f\f\f vect\mid\ vect

2 PPL Namespace Index

2.1 PPL Namespace List

Here is a list of all documented namespaces with brief descriptions:

Parma_Polyhedra_Library (The entire library is confined into this namespace)

[7](#)

3 PPL Hierarchical Index

3.1 PPL Class Hierarchy

This inheritance list is sorted roughly, but not completely, alphabetically:

Parma_Polyhedra_Library::Constraint	9
Parma_Polyhedra_Library::ConSys	10
Parma_Polyhedra_Library::ConSys::const_iterator	12
Parma_Polyhedra_Library::Generator	13
Parma_Polyhedra_Library::GenSys	15
Parma_Polyhedra_Library::GenSys::const_iterator	16
Parma_Polyhedra_Library::LinExpression	17
Parma_Polyhedra_Library::Polyhedron	19
Parma_Polyhedra_Library::Variable	26

4 PPL Compound Index

4.1 PPL Compound List

Here are the classes, structs, unions and interfaces with brief descriptions:

Parma_Polyhedra_Library::Constraint (A linear equality or inequality)	9
Parma_Polyhedra_Library::ConSys (A system of constraints)	10
Parma_Polyhedra_Library::ConSys::const_iterator ()	12
Parma_Polyhedra_Library::Generator (A line, ray or vertex)	13
Parma_Polyhedra_Library::GenSys (A system of generators)	15
Parma_Polyhedra_Library::GenSys::const_iterator ()	16
Parma_Polyhedra_Library::LinExpression (A linear expression)	17
Parma_Polyhedra_Library::Polyhedron (A convex polyhedron)	19
Parma_Polyhedra_Library::Variable (A dimension of the space)	26

5 PPL Page Index

5.1 PPL Related Pages

Here is a list of all related documentation pages:

GNU GENERAL PUBLIC LICENSE	26
GNU Free Documentation License	31

6 PPL Namespace Documentation

6.1 Parma_Polyhedra_Library Namespace Reference

The entire library is confined into this namespace.

Compounds

- class **Parma_Polyhedra_Library::Variable**
A dimension of the space.
- class **Parma_Polyhedra_Library::LinExpression**
A linear expression.
- class **Parma_Polyhedra_Library::Constraint**
A linear equality or inequality.
- class **Parma_Polyhedra_Library::ConSys**
A system of constraints.
- class **Parma_Polyhedra_Library::ConSys::const_iterator**
- class **Parma_Polyhedra_Library::Generator**
A line, ray or vertex.
- class **Parma_Polyhedra_Library::GenSys**
A system of generators.
- class **Parma_Polyhedra_Library::GenSys::const_iterator**
- class **Parma_Polyhedra_Library::Polyhedron**
A convex polyhedron.

Non-friend operators on objects of the class Variable.

- `std::ostream & operator<< (std::ostream &s, const Parma_Polyhedra_Library::Variable &v)`
Output operator.
- `bool operator< (const Variable &v, const Variable &w)`

Defines a total ordering on variables.

Non-friend operators on objects of the class **Constraint**.

- `std::ostream & operator<< (std::ostream &s, const Constraint &c)`
Output operator.

Non-friend operators on objects of the class **Generator**.

- `std::ostream & operator<< (std::ostream &s, const Generator &g)`
Output operator.

Non-friend operators on objects of the class **Polyhedron**.

- `bool operator== (const Polyhedron &x, const Polyhedron &y)`
Returns true if and only if x and y are the same polyhedron.
- `bool operator!= (const Polyhedron &x, const Polyhedron &y)`
Returns true if and only if x and y are different polyhedra.
- `bool operator< (const Polyhedron &x, const Polyhedron &y)`
Returns true if and only if x is strictly contained in y.
- `bool operator> (const Polyhedron &x, const Polyhedron &y)`
Returns true if and only if x strictly contains y.
- `bool operator>= (const Polyhedron &x, const Polyhedron &y)`
Returns true if and only if x contains y.

Enumerations

- `enum GenSys_Con_Rel { NONE_SATISFIES, ALL_SATISFY, ALL_SATURATE, SOME_SATISFY }`
Describes possible relations between a system of generators and a given constraint.

6.1.1 Enumeration Type Documentation

6.1.1.1 enum Parma_Polyhedra_Library::GenSys_Con_Rel

Enumeration values:

NONE_SATISFIES No generator satisfies the given constraint.

ALL_SATISFY All generators satisfy the given constraint, but there exists a generator not saturating it (i.e., a generator does not belong to the hyper-plane defined by the constraint.).

ALL_SATURATE All generators saturate the given constraint (i.e., they all belong to the hyper-plane defined by the constraint.).

SOME_SATISFY Some generators satisfy the given constraint (i.e., there exists both a generator satisfying the constraint and another generator which does not satisfy it.).

7 PPL Class Documentation

7.1 Parma_Polyhedra_Library::Constraint Class Reference

A linear equality or inequality.

```
#include <ppl.hh>
```

Inherits Row.

Public Methods

- **Constraint ()**
Default constructor.
- **Constraint (const Constraint &c)**
Ordinary copy-constructor.
- **~Constraint ()**
Destructor.
- **bool is_equality () const**
*Returns true if and only if *this is an equality constraint.*
- **bool is_inequality () const**
*Returns true if and only if *this is an inequality constraint.*

Friends

- Constraint **Parma_Polyhedra_Library::operator==** (const **LinExpression** &e1, const **LinExpression** &e2)
Returns the constraint $e1 = e2$.
- Constraint **Parma_Polyhedra_Library::operator==** (const **LinExpression** &e, const Integer &n)
Returns the constraint $e = n$.
- Constraint **Parma_Polyhedra_Library::operator==** (const Integer &n, const **LinExpression** &e)
Returns the constraint $n = e$.
- Constraint **Parma_Polyhedra_Library::operator>=** (const **LinExpression** &e1, const **LinExpression** &e2)
Returns the constraint $e1 \geq e2$.

- Constraint **Parma_Polyhedra_Library::operator>=** (const **LinExpression** &e, const Integer &n)
Returns the constraint $e \geq n$.
- Constraint **Parma_Polyhedra_Library::operator>=** (const Integer &n, const **LinExpression** &e)
Returns the constraint $n \geq e$.
- Constraint **Parma_Polyhedra_Library::operator<=** (const **LinExpression** &e1, const **LinExpression** &e2)
Returns the constraint $e1 \leq e2$.
- Constraint **Parma_Polyhedra_Library::operator<=** (const **LinExpression** &e, const Integer &n)
Returns the constraint $e \leq n$.
- Constraint **Parma_Polyhedra_Library::operator<=** (const Integer &n, const **LinExpression** &e)
Returns the constraint $n \leq e$.
- Constraint **Parma_Polyhedra_Library::operator>>** (const Constraint &c, unsigned int offset)
Returns the constraint c with variables renamed by adding `offset` to their Cartesian axis identifier.

7.1.1 Detailed Description

An object of the class **Constraint** (p. 9) is either:

- an equality: $\sum_{i=0}^{n-1} a_i x_i + b = 0$; or
- an inequality: $\sum_{i=0}^{n-1} a_i x_i + b \geq 0$;

where n is the dimension of the space.

How to build a constraint

Constraints are typically built by applying a relational operator to a pair of linear expressions. Available relational operators include equality (`==`) and non-strict inequalities (`>=` and `<=`). Strict inequalities (`<` and `>`) are not supported.

In the following example it is assumed that variables `x`, `y` and `z` are defined as follows:

```
Variable x(0);
Variable y(1);
Variable z(2);
```

Example

The following code builds the equality $3x + 5y - z = 0$:

```
Constraint equal(3*x + 5*y - z == 0);
```

The following code builds the constraint $4x - 2y \geq z - 13$:

```
Constraint inequal(4*x - 2*y >= z - 13);
```

7.2 Parma_Polyhedra_Library::ConSys Class Reference

A system of constraints.

```
#include <ppl.hh>
```

Inherits **Matrix**.

Public Methods

- **ConSys ()**
Default constructor: builds an empty system of constraints.
- **ConSys (const ConSys &cs)**
Ordinary copy-constructor.
- **virtual ~ConSys ()**
Destructor.
- **void insert (const Constraint &c)**
*Inserts a copy of the constraint c into *this, increasing the number of dimensions if needed.*
- **void swap (ConSys &y)**
*Swaps *this with the system of constraints y.*
- **const_iterator begin () const**
*Returns the **const_iterator** (p. 12) pointing to the first constraint, if *this is not empty; otherwise, returns the past-the-end **const_iterator** (p. 12).*
- **const_iterator end () const**
*Returns the past-the-end **const_iterator** (p. 12).*

7.2.1 Detailed Description

An object of the class **ConSys** (p. 10) is a system of constraints, i.e. a multiset of objects of the class **Constraint** (p. 9). When inserting constraints in a system, dimensions are automatically adjusted so that all the constraints in the system are defined on the same vector space.

In all the examples it is assumed that variables x and y are defined as follows:

```
Variable x(0);
Variable y(1);
```

Example 1

The following code builds a system of constraints corresponding to a square in \mathbb{R}^2 :

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x <= 3);
cs.insert(y >= 0);
cs.insert(y <= 3);
```

Example 2

The following code builds a system of constraints corresponding to a half-strip in \mathbb{R}^2 :

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x - y <= 0);
cs.insert(x - y + 1 >= 0);
```

7.3 Parma_Polyhedra_Library::ConSys::const_iterator Class Reference

#include <ppl.hh>

Inherits std::iterator.

Public Methods

- **const_iterator ()**
Default constructor.
- **const_iterator (const const_iterator &y)**
Ordinary copy-constructor.
- virtual **~const_iterator ()**
Destructor.
- **const_iterator & operator= (const const_iterator &y)**
Assignment operator.
- **const Constraint & operator * () const**
Dereference operator.
- **const Constraint * operator → () const**
Indirect member selector.
- **const_iterator & operator++ ()**
Prefix increment operator.
- **const_iterator operator++ (int)**
Postfix increment operator.
- **bool operator== (const const_iterator &y) const**
*Returns true if and only if *this and y are identical.*
- **bool operator!= (const const_iterator &y) const**
*Returns true if and only if *this and y are different.*

7.3.1 Detailed Description

A **const_iterator** (p. 12) is used to provide read-only access to each constraint contained in an object of **ConSys** (p. 10).

Example

The following code prints the system of constraints defining the polyhedron `ph`:

```
const ConSys cs = ph.constraints();
ConSys::const_iterator iend = cs.end();
for (ConSys::const_iterator i = cs.begin(); i != iend; ++i)
    cout << *i << endl;
```

7.4 Parma_Polyhedra_Library::Generator Class Reference

A line, ray or vertex.

```
#include <ppl.hh>
```

Inherits Row.

Public Types

- enum **Type**
The generator type.

Public Methods

- **Generator** ()
Default constructor.
- **Generator** (const Generator &g)
Ordinary copy-constructor.
- **~Generator** ()
Destructor.
- **Type** type () const
*Returns the generator type of *this.*

Friends

- Generator **Parma_Polyhedra_Library::line** (const **LinExpression** &e)
Returns the (bidirectional) line of direction e.
- Generator **Parma_Polyhedra_Library::ray** (const **LinExpression** &e)
Returns the (unidirectional) ray of direction e.
- Generator **Parma_Polyhedra_Library::vertex** (const **LinExpression** &e, const Integer &d=1)
Returns the vertex at e / d (note that d is an optional argument with default value 1).
Exceptions:
std::invalid_argument thrown if d is zero.

7.4.1 Detailed Description

An object of the class **Generator** (p. 13) is one of the following:

- a line $\mathbf{l} = (a_0, \dots, a_{n-1})^T$;
- a ray $\mathbf{r} = (a_0, \dots, a_{n-1})^T$;

- a vertex $\mathbf{v} = (\frac{a_0}{d}, \dots, \frac{a_{n-1}}{d})^T$;

where n is the dimension of the space.

How to build a generator.

Each type of generator is built by applying the corresponding function (`line`, `ray` or `vertex`) to a linear expression, representing a direction in the space. This means that a linear expression used to define a generator should be homogeneous and any constant term will be ignored. When defining a vertex, an optional Integer argument can be used as a common *denominator* for all the coefficients occurring in the provided linear expression; the default value for this argument is 1.

In all the following examples it is assumed that variables x , y and z are defined as follows:

```
Variable x(0);
Variable y(1);
Variable z(2);
```

Example 1

The following code builds a line with direction $x - y - z$:

```
Generator l = line(x - y - z);
```

As mentioned above, the constant term of the linear expression is not relevant. Thus, the following code has the same effect:

```
Generator l = line(x - y - z + 15);
```

Example 2

The following code builds a ray with the same direction as the line in Example 1:

```
Generator r = ray(x - y - z);
```

As is the case for lines, when specifying a ray the constant term of the linear expression is not relevant.

Example 3

The following code builds the vertex $\mathbf{v} = (1, 0, 2)^T \in \mathbb{R}^3$:

```
Generator v = vertex(1*x + 0*y + 2*z);
```

The same effect can be obtained by using the following code:

```
Generator v = vertex(x + 2*z);
```

Similarly, the origin $\mathbf{0} \in \mathbb{R}^3$ can be defined using either one of the following lines of code:

```
Generator origin1 = vertex(0*x + 0*y + 0*z);
Generator origin2 = vertex(0*z);
```

Note however that the following line would have defined a different vertex, namely $\mathbf{0} \in \mathbb{R}^2$:

```
Generator origin3 = vertex(0*y);
```

Example 4

The vertex \mathbf{v} specified in Example 3 above can also be obtained with the following code, where we provide a non-default value for the denominator argument:

```
Generator v = vertex(2*x + 0*y + 4*z, 2);
```

Obviously, the denominator can be usefully exploited to specify vertices having some non-integer (but rational) coordinates. For instance, the vertex $\mathbf{w} = (-1.5, 3.2, 2.1)^T \in \mathbb{R}^3$ can be specified by the following code:

```
Generator w = vertex(-15*x + 32*y + 21*z, 10);
```

If a zero denominator is provided, an exception is thrown.

7.5 Parma_Polyhedra_Library::GenSys Class Reference

A system of generators.

```
#include <ppl.hh>
```

Inherits Matrix.

Public Methods

- **GenSys ()**
Default constructor: builds an empty system of generators.
- **GenSys (const GenSys &gs)**
Ordinary copy-constructor.
- **virtual ~GenSys ()**
Destructor.
- **void insert (const Generator &g)**
*Inserts a copy of the generator g into $*this$, increasing the number of dimensions if needed.*
- **void swap (GenSys &y)**
*Swaps $*this$ with the system of generators y .*
- **const_iterator begin () const**
*Returns the **const_iterator** (p. 16) pointing to the first generator, if $*this$ is not empty; otherwise, returns the past-the-end **const_iterator** (p. 16).*
- **const_iterator end () const**
*Returns the past-the-end **const_iterator** (p. 16).*

7.5.1 Detailed Description

An object of the class **GenSys** (p. 15) is a system of generators, i.e. a multiset of objects of the class **Generator** (p. 13) (lines, rays and vertices). When inserting generators in a system, dimensions are automatically adjusted so that all the generators in the system are defined on the same vector space. A system of generators which is meant to define a non-empty polyhedron must include at least one vertex, even if the polyhedron has no “proper” vertices: the reason is that lines and rays need a supporting point (they only specify directions).

In all the examples it is assumed that variables x and y are defined as follows:

```
Variable x(0);
Variable y(1);
```

Example 1

The following code defines the line having the same direction as the x axis (i.e., the first Cartesian axis) in \mathbb{R}^2 :

```
GenSys gs;
gs.insert(line(x + 0*y));
```


As said above, this system of generators corresponds to an empty polyhedron, because the line has no supporting point. To define a system of generators indeed corresponding to the x axis, one can add the following code which inserts the origin of the space as a vertex:

```
gs.insert(vertex(0*x + 0*y));
```

Since dimensions are automatically adjusted, the following code obtains the same effect:

```
gs.insert(vertex(0*x));
```

In contrast, if we had added the following code, we would have defined a line parallel to the x axis and including the point $(0, 1)^T \in \mathbb{R}^2$.

```
gs.insert(vertex(0*x + 1*y));
```

Example 2

The following code builds a ray having the same direction as the positive part of the x axis in \mathbb{R}^2 :

```
GenSys gs;
gs.insert(ray(x + 0*y));
```

To define a system of generators indeed corresponding to the set

$$\{(x, 0)^T \in \mathbb{R}^2 \mid x \geq 0\},$$

one just has to add the origin:

```
gs.insert(vertex(0*x + 0*y));
```

Example 3

The following code builds a system of generators having four vertices and corresponding to a square in \mathbb{R}^2 (the same as Example 1 for the system of constraints):

```
GenSys gs;
gs.insert(vertex(0*x + 0*y));
gs.insert(vertex(0*x + 3*y));
gs.insert(vertex(3*x + 0*y));
gs.insert(vertex(3*x + 3*y));
```

Example 4

The following code builds a system of generators having two vertices and a ray, corresponding to a half-strip in \mathbb{R}^2 (the same as Example 2 for the system of constraints):

```
GenSys gs;
gs.insert(vertex(0*x + 0*y));
gs.insert(vertex(0*x + 1*y));
gs.insert(ray(x - y));
```

7.6 Parma_Polyhedra_Library::GenSys::const_iterator Class Reference

```
#include <ppl.hh>
```

Inherits std::iterator.

Public Methods

- **const_iterator** ()
Default constructor.
- **const_iterator** (const const_iterator &y)
Ordinary copy-constructor.
- virtual **~const_iterator** ()
Destructor.
- const_iterator & **operator=** (const const_iterator &y)
Assignment operator.
- const **Generator** & **operator *** () const
Dereference operator.
- const **Generator** * **operator** → () const
Indirect member selector.
- const_iterator & **operator++** ()
Prefix increment operator.
- const_iterator **operator++** (int)
Postfix increment operator.
- bool **operator==** (const const_iterator &y) const
*Returns true if and only if *this and y are identical.*
- bool **operator!=** (const const_iterator &y) const
*Returns true if and only if *this and y are different.*

7.6.1 Detailed Description

A **const_iterator** (p. 16) is used to provide read-only access to each generator contained in an object of **GenSys** (p. 15).

Example

The following code prints the system of generators of the polyhedron `ph`:

```
const GenSys gs = ph.generators();
GenSys::const_iterator iend = gs.end();
for (GenSys::const_iterator i = gs.begin(); i != iend; ++i)
    cout << *i << endl;
```

7.7 Parma_Polyhedra_Library::LinExpression Class Reference

A linear expression.

```
#include <ppl.hh>
```

Inherits Row.

Public Methods

- **LinExpression ()**
Default constructor.
- **LinExpression (const LinExpression &e)**
Ordinary copy-constructor.
- **virtual ~LinExpression ()**
Destructor.
- **LinExpression (const Integer &n)**
Constructor: builds the linear expression corresponding to the inhomogeneous term n .
- **LinExpression (const Variable &v)**
Constructor: builds the linear expression corresponding to the variable v .

Friends

- **LinExpression Parma_Polyhedra_Library::operator+ (const LinExpression &e1, const LinExpression &e2)**
Returns the linear expression $e1 + e2$.
- **LinExpression Parma_Polyhedra_Library::operator+ (const Integer &n, const LinExpression &e)**
Returns the linear expression $n + e$.
- **LinExpression Parma_Polyhedra_Library::operator+ (const LinExpression &e, const Integer &n)**
Returns the linear expression $e + n$.
- **LinExpression Parma_Polyhedra_Library::operator- (const LinExpression &e)**
Returns the linear expression $- e$.
- **LinExpression Parma_Polyhedra_Library::operator- (const LinExpression &e1, const LinExpression &e2)**
Returns the linear expression $e1 - e2$.
- **LinExpression Parma_Polyhedra_Library::operator- (const Integer &n, const LinExpression &e)**
Returns the linear expression $n - e$.
- **LinExpression Parma_Polyhedra_Library::operator- (const LinExpression &e, const Integer &n)**
Returns the linear expression $e - n$.
- **LinExpression Parma_Polyhedra_Library::operator* (const Integer &n, const LinExpression &e)**
*Returns the linear expression $n * e$.*
- **LinExpression Parma_Polyhedra_Library::operator* (const LinExpression &e, const Integer &n)**
*Returns the linear expression $e * n$.*

- `LinExpression & Parma_Polyhedra_Library::operator+= (LinExpression &e1, const LinExpression &e2)`
Returns the linear expression $e1 + e2$ and assigns it to $e1$.
- `LinExpression & Parma_Polyhedra_Library::operator+= (LinExpression &e, const Variable &v)`
Returns the linear expression $e + v$ and assigns it to e .
- `LinExpression & Parma_Polyhedra_Library::operator+= (LinExpression &e, const Integer &n)`
Returns the linear expression $e + n$ and assigns it to e .

7.7.1 Detailed Description

An object of the class **LinExpression** (p. 17) represents the linear expression

$$\sum_{i=0}^{n-1} a_i x_i + b$$

where n is the dimension of the space, each a_i is the integer coefficient of the i -th variable x_i and b is the integer for the inhomogeneous term.

How to build a linear expression.

Linear expressions are the basic blocks for defining both constraints (i.e., linear equalities or inequalities) and generators (i.e., lines, rays and vertices). The following functions provide a convenient interface for building a complex linear expression starting from simpler ones (or even from objects of the classes **Variable** (p. 26) and **Integer**). Available operators include unary negation, binary addition and subtraction, as well as multiplication by an **Integer**.

Example

The following code builds the linear expression $4x - 2y - z + 14$:

```
LinExpression e = 4*x - 2*y - z + 14;
```

Another way to build the same linear expression is:

```
LinExpression e1 = 4*x;
LinExpression e2 = 2*y;
LinExpression e3 = z;
LinExpression e = LinExpression(14);
e += e1 - e2 - e3;
```

7.8 Parma_Polyhedra_Library::Polyhedron Class Reference

A convex polyhedron.

```
#include <ppl.hh>
```

Public Types

- enum **Degenerate_Kind** { **ZERO_DIMENSIONAL**, **EMPTY** }
Kinds of degenerate polyhedra.

Public Methods

- **Polyhedron (Degenerate_Kind kind=ZERO_DIMENSIONAL)**
Builds the zero-dimensional, universe polyhedron, if kind is ZERO_DIMENSIONAL (the default); otherwise (i.e., if kind is EMPTY) builds an empty polyhedron.
- **Polyhedron (const Polyhedron &y)**
Ordinary copy-constructor.
- **Polyhedron (size_t num_dimensions)**
Builds the universe polyhedron of dimension num_dimensions.
- **Polyhedron (ConSys &cs)**
Builds a polyhedron from a system of constraints.
Parameters:
cs The system of constraints defining the polyhedron. It is not declared const because it can be modified.
- **Polyhedron (GenSys &gs)**
Builds a polyhedron from a system of generators.
Parameters:
gs The system of generators defining the polyhedron. It is not declared const because it can be modified.
Exceptions:
std::invalid_argument thrown if the system of generators has no vertex.
- **Polyhedron & operator= (const Polyhedron &y)**
The assignment operator.
- **size_t num_dimensions () const**
Returns the dimension of the polyhedron.
- **void intersection_assign (const Polyhedron &y)**
*Intersects *this with polyhedron y and assigns the result to *this.*
Exceptions:
*std::invalid_argument thrown if *this and y have different dimension.*
- **void convex_hull_assign (const Polyhedron &y)**
*Assigns the convex hull of *this \cup y to *this.*
Exceptions:
*std::invalid_argument thrown if *this and y have different dimension.*
- **void convex_hull_assign_lazy (const Polyhedron &y)**
*Assigns the convex hull of *this \cup y to *this, without minimizing the result.*
Exceptions:
*std::invalid_argument thrown if *this and y have different dimension.*
- **GenSys_Con_Rel satisfies (const Constraint &c)**
*Returns the relation between the generators of *this and the constraint c.*
Exceptions:
*std::invalid_argument thrown if *this and constraint c have different dimension.*

- **bool includes** (const **Generator** &g)

*Tests the inclusion of the generator g in the polyhedron *this.*

Exceptions:

std::invalid_argument thrown if *this and constraint g have different dimension.
- **void widening_assign** (const Polyhedron &y)

*Computes the widening between *this and y and assigns the result to *this.*

Parameters:

y The polyhedron that must be contained in *this.

Exceptions:

std::invalid_argument thrown if *this and y have different dimension.
- **bool limited_widening_assign** (const Polyhedron &y, **ConSys** &cs)

*Limits the widening between *this and y by cs and assigns the result to *this.*

Parameters:

y The polyhedron that must be contained in *this.
cs The system of constraints that limits the widened polyhedron. It is not declared const because it can be modified.

Returns:

true if the resulting polyhedron is not empty false otherwise.

Exceptions:

std::invalid_argument thrown if *this, y and cs have different dimension.
- **const ConSys & constraints** () const

Returns the system of constraints.

Exceptions:

std::invalid_argument thrown if *this is empty.
- **const GenSys & generators** () const

Returns the system of generators.

Exceptions:

std::invalid_argument thrown if *this is zero-dimensional.
- **void insert** (const **Constraint** &c)

*Inserts a new constraint c into the system of constraints of *this.*
- **void insert** (const **Generator** &g)

*Inserts a new generator g into the system of generators of *this.*
- **void assign_variable** (const **Variable** &v, const **LinExpression** &expr, const Integer &denominator=1)

Assigns an affine expression to the specified variable.

Parameters:

v The variable to which the affine expression is assigned.
expr The numerator of the affine expression.
denominator The denominator of the affine expression (optional argument with default value 1.)

Exceptions:

std::invalid_argument thrown if denominator is zero or if expr and *this have different dimension or if v is not a variable of the polyhedron.
- **void substitute_variable** (const **Variable** &v, const **LinExpression** &expr, const Integer &denominator=1)

Substitutes an affine expression for the specified variable.

Parameters:

v The variable to which the affine expression is substituted.
expr The numerator of the affine expression.
denominator The denominator of the affine expression (optional argument with default value 1.)

Exceptions:

std::invalid_argument thrown if denominator is zero or if *expr* and **this* have different dimension or if *v* is not a variable of the polyhedron.

- **bool OK** (bool check_not_empty=true) const

Checks if all the invariants are satisfied.

Parameters:

check_not_empty true if it must be checked whether the system of constraint is satisfiable.

Returns:

true if the polyhedron satisfies all the invariants stated in the PPL, false otherwise.

- **void add_dimensions_and_embed** (size_t dim)

Adds new dimensions and embeds the old polyhedron in the new space.

Parameters:

dim The number of dimensions to add.

- **void add_dimensions_and_project** (size_t dim)

Adds new dimensions to the polyhedron and does not embed it in the new space.

Parameters:

dim The number of dimensions to add.

- **void remove_dimensions** (const std::set< **Variable** > &to_be_removed)

Removes the specified dimensions.

Parameters:

to_be_removed The set of variables to remove.

- **bool add_constraints** (ConSys &cs)

Adds the specified constraints and computes a new polyhedron.

Parameters:

cs The constraints that will be added to the current system of constraints. This parameter is not declared const because it can be modified.

Returns:

false if the resulting polyhedron is empty.

Exceptions:

std::invalid_argument thrown if **this* and *cs* have different dimension.

- **void add_constraints_lazy** (ConSys &cs)

Adds the specified constraints without minimizing.

Parameters:

cs The constraints that will be added to the current system of constraints. This parameter is not declared const because it can be modified.

Exceptions:

std::invalid_argument thrown if **this* and *cs* have different dimension.

- **void add_generators** (GenSys &gs)

Adds the specified generators.

Parameters:

gs The generators that will be added to the current system of generators. The parameter is not declared const because it can be modified.

Exceptions:

std::invalid_argument thrown if **this* and *gs* have different dimension.

- **bool check_empty () const**
Returns true if and only if the polyhedron is empty.
- **bool check_universe () const**
*Returns true if *this is a universe polyhedron.*
- **void swap (Polyhedron &y)**
*Swaps *this with polyhedron y.*
- **bool is_empty () const**
*Returns true if and only if *this is an empty polyhedron.*
- **bool is_zero_dim () const**
*Returns true if and only if *this is a zero-dimensional polyhedron.*

Friends

- **bool Parma_Polyhedra_Library::operator<= (const Polyhedron &x, const Polyhedron &y)**
Returns true if and only if polyhedron x is contained in polyhedron y.
- **std::ostream & Parma_Polyhedra_Library::operator<< (std::ostream &s, const Polyhedron &p)**
Output operator.
- **std::istream & Parma_Polyhedra_Library::operator>> (std::istream &s, Polyhedron &p)**
Input operator.

7.8.1 Detailed Description

An object of the class **Polyhedron** (p. 19) represents a convex polyhedron in the space \mathbb{R}^n .

A polyhedron can be specified as either a finite system of constraints or a finite system of generators (see Minkowski's theorem in the Introduction). So, it is possible to obtain one system from the other. That is, if we know the system of constraints, we can obtain from this the system of generators that define the same polyhedron and vice versa. These systems can contain some redundant members: in this case we say that they are not in the minimal form.

In all the examples it is assumed that variables **x** and **y** are defined (where they are used) as follows:

```
Variable x(0);
Variable y(1);
```

Example 1

The following code builds a polyhedron corresponding to a square in \mathbb{R}^2 , given as a system of constraints:

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x <= 3);
cs.insert(y >= 0);
cs.insert(y <= 3);
Polyhedron ph(cs);
```


The following code builds the same polyhedron as above, but starting from a system of generators specifying the four vertices of the square:

```
GenSys gs;
gs.insert(vertex(0*x + 0*y));
gs.insert(vertex(0*x + 3*y));
gs.insert(vertex(3*x + 0*y));
gs.insert(vertex(3*x + 3*y));
Polyhedron ph(gs);
```

Example 2

The following code builds an unbounded polyhedron corresponding to a half-strip in \mathbb{R}^2 , given as a system of constraints:

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x - y <= 0);
cs.insert(x - y + 1 >= 0);
Polyhedron ph(cs);
```

The following code builds the same polyhedron as above, but starting from the system of generators specifying the two vertices of the polyhedron and one ray:

```
GenSys gs;
gs.insert(vertex(0*x + 0*y));
gs.insert(vertex(0*x + y));
gs.insert(ray(x - y));
Polyhedron ph(gs);
```

Example 3

The following code builds the polyhedron corresponding to an half-plane in \mathbb{R}^2 , by adding a single constraint to the universe polyhedron:

```
Polyhedron ph;
ph.insert(y >= 0);
```

The following code builds the same polyhedron as above, but starting from a system of generators specifying a vertex, a ray and a line.

```
Polyhedron ph;
ph.insert(vertex(0*x + 0*y));
ph.insert(ray(0*x + y));
ph.insert(line(x + 0*y));
```

In this last case, it is important to note that: even if this polyhedron has no real vertex, we must add one, because otherwise the polyhedron is considered empty.

Example 4

The following code shows the use of the function `add_dimensions_and_embed`:

```
Polyhedron ph;
ph.insert(x == 2);
ph.add_dimensions_and_embed(1);
```

We start with the universe polyhedron in the 0-dimensional space. Then we add a single equality constraint, thus obtaining the polyhedron corresponding to the singleton set $\{2\} \subseteq \mathbb{R}$. After the last line of code, the resulting polyhedron is

$$\{ (2, x_1)^T \in \mathbb{R}^2 \mid x_1 \in \mathbb{R} \}.$$

Example 5

The following code shows the use of the function `add_dimensions_and_project`:

```
Polyhedron ph;
ph.insert(x == 2);
ph.add_dimensions_and_project(1);
```

The first two lines of code are the same as in Example 4 for `add_dimensions_and_embed`. After the last line of code, the resulting polyhedron is the singleton set $\{(2, 0)^T\} \subseteq \mathbb{R}^2$.

Example 6

The following code shows the use of the function `assign_variable`:

```
Polyhedron ph;
ph.insert(vertex(0*x + 0*y));
ph.insert(vertex(0*x + 3*y));
ph.insert(vertex(3*x + 0*y));
ph.insert(vertex(3*x + 3*y));
LinExpression coeff = x + 0*y + 4;
ph.assign_variable(x, coeff);
```

In this example the starting polyhedron is a square in \mathbb{R}^2 , the considered variable is x and the affine expression is $x + 4$. The resulting polyhedron is the same square translated towards right. Moreover, if the affine transformation for the same variable x is $x + y$:

```
LinExpression coeff = x + y;
```

the resulting polyhedron is a parallelogram with the height equal to the side of the square and the oblique sides parallel to the line $x - y$. Instead, if we do not use an invertible transformation for the same variable; for example, the affine expression y :

```
LinExpression coeff = 0*x + y;
```

the resulting polyhedron is a diagonal of the square.

Example 7

The following code shows the use of the function `substitute_variable`:

```
Polyhedron ph;
ph.insert(x >= 0);
ph.insert(x <= 3);
ph.insert(y >= 0);
ph.insert(y <= 3);
LinExpression coeff = x + 0*y + 4;
ph.substitute_variable(x, coeff);
```

In this example the starting polyhedron, `var` and the affine expression and the denominator are the same as in Example 6, while the resulting polyhedron is again the same square but translated towards left. Moreover, if the affine transformation for x is $x + y$

```
LinExpression coeff = x + y;
```

the resulting polyhedron is a parallelogram with the height equal to the side of the square and the oblique sides parallel to the line $x + y$. Instead, if we do not use an invertible transformation for the same variable x , for example, the affine expression y :

```
LinExpression coeff = 0*x + y;
```

the resulting polyhedron is a line that corresponds to the y axis.

7.8.2 Member Enumeration Documentation

7.8.2.1 enum Parma_Polyhedra_Library::Polyhedron::Degenerate_Kind

Enumeration values:

ZERO_DIMENSIONAL The full polyhedron in \mathbb{R}^0 , i.e., a singleton.

EMPTY The empty polyhedron, i.e., the empty set.

7.9 Parma_Polyhedra_Library::Variable Class Reference

A dimension of the space.

```
#include <ppl.hh>
```

Public Methods

- **Variable** (unsigned int id)
Constructor: id is the index of the Cartesian axis.
- unsigned int **id** () const
Returns the index of the Cartesian axis.

7.9.1 Detailed Description

An object of the class **Variable** (p. 26) represents a dimension of the space, that is one of the Cartesian axes. Variables are used as base blocks in order to build more complex linear expressions. Each variable is identified by a non-negative integer, representing the index of the corresponding Cartesian axis (the first axis has index 0).

Note that the “meaning” of an object of the class **Variable** (p. 26) is completely specified by the integer index provided to its constructor: be careful not to be misled by C++ language variable names. For instance, in the following example the linear expressions `e1` and `e2` are equivalent, since the two variables `x` and `z` denote the same Cartesian axis.

```
Variable x(0);
Variable y(1);
Variable z(0);
LinExpression e1 = x + y;
LinExpression e2 = y + z;
```

8 PPL Page Documentation

8.1 GNU GENERAL PUBLIC LICENSE

Version 2, June 1991

Copyright (C) 1989, 1991 Free Software Foundation, Inc.
59 Temple Place - Suite 330, Boston, MA 02111-1307, USA

Everyone is permitted to copy and distribute verbatim copies
of this license document, but changing it is not allowed.

Preamble

The licenses for most software are designed to take away your freedom to share and change it. By contrast, the GNU General Public License is intended to guarantee your freedom to share and change free software—to make sure the software is free for all its users. This General Public License applies to most of the Free Software Foundation’s software and to any other program whose authors commit to using it. (Some other Free Software Foundation software is covered by the GNU Library General Public License instead.) You can apply it to your programs, too.

When we speak of free software, we are referring to freedom, not price. Our General Public Licenses are designed to make sure that you have the freedom to distribute copies of free software (and charge for this service if you wish), that you receive source code or can get it if you want it, that you can change the software or use pieces of it in new free programs; and that you know you can do these things.

To protect your rights, we need to make restrictions that forbid anyone to deny you these rights or to ask you to surrender the rights. These restrictions translate to certain responsibilities for you if you distribute copies of the software, or if you modify it.

For example, if you distribute copies of such a program, whether gratis or for a fee, you must give the recipients all the rights that you have. You must make sure that they, too, receive or can get the source code. And you must show them these terms so they know their rights.

We protect your rights with two steps: (1) copyright the software, and (2) offer you this license which gives you legal permission to copy, distribute and/or modify the software.

Also, for each author’s protection and ours, we want to make certain that everyone understands that there is no warranty for this free software. If the software is modified by someone else and passed on, we want its recipients to know that what they have is not the original, so that any problems introduced by others will not reflect on the original authors’ reputations.

Finally, any free program is threatened constantly by software patents. We wish to avoid the danger that redistributors of a free program will individually obtain patent licenses, in effect making the program proprietary. To prevent this, we have made it clear that any patent must be licensed for everyone’s free use or not licensed at all.

The precise terms and conditions for copying, distribution and modification follow.

TERMS AND CONDITIONS FOR COPYING, DISTRIBUTION AND MODIFICATION

0. This License applies to any program or other work which contains a notice placed by the copyright holder saying it may be distributed under the terms of this General Public License. The “Program”, below, refers to any such program or work, and a “work based on the Program” means either the Program or any derivative work under copyright law: that is to say, a work containing the Program or a portion of it, either verbatim or with modifications and/or translated into another language. (Hereinafter, translation is included without limitation in the term “modification”.) Each licensee is addressed as “you”.

Activities other than copying, distribution and modification are not covered by this License; they are outside its scope. The act of running the Program is not restricted, and the output from the Program is covered only if its contents constitute a work based on the Program (independent of having been made by running the Program). Whether that is true depends on what the Program does.

1. You may copy and distribute verbatim copies of the Program’s source code as you receive it, in any medium, provided that you conspicuously and appropriately publish on each copy an appropriate copyright notice and disclaimer of warranty; keep intact all the notices that refer to this License and to the absence of any warranty; and give any other recipients of the Program a copy of this License along with the Program.

You may charge a fee for the physical act of transferring a copy, and you may at your option offer warranty

protection in exchange for a fee.

2. You may modify your copy or copies of the Program or any portion of it, thus forming a work based on the Program, and copy and distribute such modifications or work under the terms of Section 1 above, provided that you also meet all of these conditions:

- **a)** You must cause the modified files to carry prominent notices stating that you changed the files and the date of any change.
- **b)** You must cause any work that you distribute or publish, that in whole or in part contains or is derived from the Program or any part thereof, to be licensed as a whole at no charge to all third parties under the terms of this License.
- **c)** If the modified program normally reads commands interactively when run, you must cause it, when started running for such interactive use in the most ordinary way, to print or display an announcement including an appropriate copyright notice and a notice that there is no warranty (or else, saying that you provide a warranty) and that users may redistribute the program under these conditions, and telling the user how to view a copy of this License. (Exception: if the Program itself is interactive but does not normally print such an announcement, your work based on the Program is not required to print an announcement.)

These requirements apply to the modified work as a whole. If identifiable sections of that work are not derived from the Program, and can be reasonably considered independent and separate works in themselves, then this License, and its terms, do not apply to those sections when you distribute them as separate works. But when you distribute the same sections as part of a whole which is a work based on the Program, the distribution of the whole must be on the terms of this License, whose permissions for other licensees extend to the entire whole, and thus to each and every part regardless of who wrote it.

Thus, it is not the intent of this section to claim rights or contest your rights to work written entirely by you; rather, the intent is to exercise the right to control the distribution of derivative or collective works based on the Program.

In addition, mere aggregation of another work not based on the Program with the Program (or with a work based on the Program) on a volume of a storage or distribution medium does not bring the other work under the scope of this License.

3. You may copy and distribute the Program (or a work based on it, under Section 2) in object code or executable form under the terms of Sections 1 and 2 above provided that you also do one of the following:

- **a)** Accompany it with the complete corresponding machine-readable source code, which must be distributed under the terms of Sections 1 and 2 above on a medium customarily used for software interchange; or,
- **b)** Accompany it with a written offer, valid for at least three years, to give any third party, for a charge no more than your cost of physically performing source distribution, a complete machine-readable copy of the corresponding source code, to be distributed under the terms of Sections 1 and 2 above on a medium customarily used for software interchange; or,
- **c)** Accompany it with the information you received as to the offer to distribute corresponding source code. (This alternative is allowed only for noncommercial distribution and only if you received the program in object code or executable form with such an offer, in accord with Subsection b above.)

The source code for a work means the preferred form of the work for making modifications to it. For an executable work, complete source code means all the source code for all modules it contains, plus any associated interface definition files, plus the scripts used to control compilation and installation of the executable. However, as a special exception, the source code distributed need not include anything that is normally distributed (in either source or binary form) with the major components (compiler, kernel, and so on) of the operating system on which the executable runs, unless that component itself accompanies the executable.

If distribution of executable or object code is made by offering access to copy from a designated place, then offering equivalent access to copy the source code from the same place counts as distribution of the source code, even though third parties are not compelled to copy the source along with the object code.

4. You may not copy, modify, sublicense, or distribute the Program except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense or distribute the Program is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

5. You are not required to accept this License, since you have not signed it. However, nothing else grants you permission to modify or distribute the Program or its derivative works. These actions are prohibited by law if you do not accept this License. Therefore, by modifying or distributing the Program (or any work based on the Program), you indicate your acceptance of this License to do so, and all its terms and conditions for copying, distributing or modifying the Program or works based on it.

6. Each time you redistribute the Program (or any work based on the Program), the recipient automatically receives a license from the original licensor to copy, distribute or modify the Program subject to these terms and conditions. You may not impose any further restrictions on the recipients' exercise of the rights granted herein. You are not responsible for enforcing compliance by third parties to this License.

7. If, as a consequence of a court judgment or allegation of patent infringement or for any other reason (not limited to patent issues), conditions are imposed on you (whether by court order, agreement or otherwise) that contradict the conditions of this License, they do not excuse you from the conditions of this License. If you cannot distribute so as to satisfy simultaneously your obligations under this License and any other pertinent obligations, then as a consequence you may not distribute the Program at all. For example, if a patent license would not permit royalty-free redistribution of the Program by all those who receive copies directly or indirectly through you, then the only way you could satisfy both it and this License would be to refrain entirely from distribution of the Program.

If any portion of this section is held invalid or unenforceable under any particular circumstance, the balance of the section is intended to apply and the section as a whole is intended to apply in other circumstances.

It is not the purpose of this section to induce you to infringe any patents or other property right claims or to contest validity of any such claims; this section has the sole purpose of protecting the integrity of the free software distribution system, which is implemented by public license practices. Many people have made generous contributions to the wide range of software distributed through that system in reliance on consistent application of that system; it is up to the author/donor to decide if he or she is willing to distribute software through any other system and a licensee cannot impose that choice.

This section is intended to make thoroughly clear what is believed to be a consequence of the rest of this License.

8. If the distribution and/or use of the Program is restricted in certain countries either by patents or by copyrighted interfaces, the original copyright holder who places the Program under this License may add an explicit geographical distribution limitation excluding those countries, so that distribution is permitted only in or among countries not thus excluded. In such case, this License incorporates the limitation as if written in the body of this License.

9. The Free Software Foundation may publish revised and/or new versions of the General Public License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns.

Each version is given a distinguishing version number. If the Program specifies a version number of this License which applies to it and "any later version", you have the option of following the terms and conditions either of that version or of any later version published by the Free Software Foundation. If the Program does not specify a version number of this License, you may choose any version ever published by the Free Software Foundation.

10. If you wish to incorporate parts of the Program into other free programs whose distribution conditions are different, write to the author to ask for permission. For software which is copyrighted by the Free Software Foundation, write to the Free Software Foundation; we sometimes make exceptions for this. Our decision will be guided by the two goals of preserving the free status of all derivatives of our free software and of promoting the sharing and reuse of software generally.

NO WARRANTY

11. BECAUSE THE PROGRAM IS LICENSED FREE OF CHARGE, THERE IS NO WARRANTY FOR THE PROGRAM, TO THE EXTENT PERMITTED BY APPLICABLE LAW. EXCEPT WHEN OTHERWISE STATED IN WRITING THE COPYRIGHT HOLDERS AND/OR OTHER PARTIES PROVIDE THE PROGRAM "AS IS" WITHOUT WARRANTY OF ANY KIND, EITHER EXPRESSED OR IMPLIED, INCLUDING, BUT NOT LIMITED TO, THE IMPLIED WARRANTIES OF MERCHANTABILITY AND FITNESS FOR A PARTICULAR PURPOSE. THE ENTIRE RISK AS TO THE QUALITY AND PERFORMANCE OF THE PROGRAM IS WITH YOU. SHOULD THE PROGRAM PROVE DEFECTIVE, YOU ASSUME THE COST OF ALL NECESSARY SERVICING, REPAIR OR CORRECTION.

12. IN NO EVENT UNLESS REQUIRED BY APPLICABLE LAW OR AGREED TO IN WRITING WILL ANY COPYRIGHT HOLDER, OR ANY OTHER PARTY WHO MAY MODIFY AND/OR REDISTRIBUTE THE PROGRAM AS PERMITTED ABOVE, BE LIABLE TO YOU FOR DAMAGES, INCLUDING ANY GENERAL, SPECIAL, INCIDENTAL OR CONSEQUENTIAL DAMAGES ARISING OUT OF THE USE OR INABILITY TO USE THE PROGRAM (INCLUDING BUT NOT LIMITED TO LOSS OF DATA OR DATA BEING RENDERED INACCURATE OR LOSSES SUSTAINED BY YOU OR THIRD PARTIES OR A FAILURE OF THE PROGRAM TO OPERATE WITH ANY OTHER PROGRAMS), EVEN IF SUCH HOLDER OR OTHER PARTY HAS BEEN ADVISED OF THE POSSIBILITY OF SUCH DAMAGES.

END OF TERMS AND CONDITIONS

How to Apply These Terms to Your New Programs

If you develop a new program, and you want it to be of the greatest possible use to the public, the best way to achieve this is to make it free software which everyone can redistribute and change under these terms.

To do so, attach the following notices to the program. It is safest to attach them to the start of each source file to most effectively convey the exclusion of warranty; and each file should have at least the "copyright" line and a pointer to where the full notice is found.

```
one line to give the program's name and an idea of what it does.
Copyright (C) yyyy  name of author
```

```
This program is free software; you can redistribute it and/or
modify it under the terms of the GNU General Public License
as published by the Free Software Foundation; either version 2
of the License, or (at your option) any later version.
```

```
This program is distributed in the hope that it will be useful,
but WITHOUT ANY WARRANTY; without even the implied warranty of
MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE.  See the
GNU General Public License for more details.
```

```
You should have received a copy of the GNU General Public License
along with this program; if not, write to the Free Software
Foundation, Inc., 59 Temple Place - Suite 330, Boston, MA 02111-1307, USA.
```


Also add information on how to contact you by electronic and paper mail.

If the program is interactive, make it output a short notice like this when it starts in an interactive mode:

```
Gnomovision version 69, Copyright (C) year name of author
Gnomovision comes with ABSOLUTELY NO WARRANTY; for details
type `show w'. This is free software, and you are welcome
to redistribute it under certain conditions; type `show c'
for details.
```

The hypothetical commands `<SAMP>'show w'</SAMP>` and `<SAMP>'show c'</SAMP>` should show the appropriate parts of the General Public License. Of course, the commands you use may be called something other than `<SAMP>'show w'</SAMP>` and `<SAMP>'show c'</SAMP>`; they could even be mouse-clicks or menu items—whatever suits your program.

You should also get your employer (if you work as a programmer) or your school, if any, to sign a “copyright disclaimer” for the program, if necessary. Here is a sample; alter the names:

```
Yoyodyne, Inc., hereby disclaims all copyright
interest in the program 'Gnomovision'
(which makes passes at compilers) written
by James Hacker.
```

```
signature of Ty Coon, 1 April 1989
Ty Coon, President of Vice
```

This General Public License does not permit incorporating your program into proprietary programs. If your program is a subroutine library, you may consider it more useful to permit linking proprietary applications with the library. If this is what you want to do, use the GNU Library General Public License instead of this License.

8.2 GNU Free Documentation License

Version 1.1, March 2000

```
Copyright (C) 2000 Free Software Foundation, Inc.
59 Temple Place, Suite 330, Boston, MA 02111-1307 USA
Everyone is permitted to copy and distribute verbatim copies
of this license document, but changing it is not allowed.
```

0. PREAMBLE

The purpose of this License is to make a manual, textbook, or other written document “free” in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondly, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of “copyleft”, which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work,

regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. The "Document", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "you".

A "Modified Version" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A "Secondary Section" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (For example, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The "Invariant Sections" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License.

The "Cover Texts" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License.

A "Transparent" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, whose contents can be viewed and edited directly and straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup has been designed to thwart or discourage subsequent modification by readers is not Transparent. A copy that is not "Transparent" is called "Opaque".

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML designed for human modification. Opaque formats include PostScript, PDF, proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML produced by some word processors for output purposes only.

The "Title Page" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work's title, preceding the beginning of the body of the text.

2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

3. COPYING IN QUANTITY

If you publish printed copies of the Document numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers

must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a publicly-accessible computer-network location containing a complete Transparent copy of the Document, free of added material, which the general network-using public has access to download anonymously at no charge using public-standard network protocols. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

- **A.** Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
- **B.** List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has less than five).
- **C.** State on the Title page the name of the publisher of the Modified Version, as the publisher.
- **D.** Preserve all the copyright notices of the Document.
- **E.** Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
- **F.** Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
- **G.** Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
- **H.** Include an unaltered copy of this License.
- **I.** Preserve the section entitled "History", and its title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.
- **J.** Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.
- **K.** In any section entitled "Acknowledgements" or "Dedications", preserve the section's title, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.

- **L.** Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.
- **M.** Delete any section entitled "Endorsements". Such a section may not be included in the Modified Version.
- **N.** Do not retitle any existing section as "Endorsements" or to conflict in title with any Invariant Section.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections entitled "History" in the various original documents, forming one section entitled "History"; likewise combine any sections entitled "Acknowledgements", and any sections entitled "Dedications". You must delete all sections entitled "Endorsements."

6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, does not as a whole count as a Modified Version of the Document, provided no compilation copyright is claimed for the compilation. Such a compilation is called an "aggregate", and this License does not apply to the other self-contained works thus compiled

with the Document, on account of their being thus compiled, if they are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one quarter of the entire aggregate, the Document's Cover Texts may be placed on covers that surround only the Document within the aggregate. Otherwise they must appear on covers around the whole aggregate.

8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License provided that you also include the original English version of this License. In case of a disagreement between the translation and the original English version of this License, the original English version will prevail.

9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided for under this License. Any other attempt to copy, modify, sublicense or distribute the Document is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See <http://www.gnu.org/copyleft/>.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation.

How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

```
Copyright (c) YEAR YOUR NAME.
Permission is granted to copy, distribute and/or modify this document
under the terms of the GNU Free Documentation License, Version 1.1
or any later version published by the Free Software Foundation;
with the Invariant Sections being LIST THEIR TITLES, with the
Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.
A copy of the license is included in the section entitled "GNU
Free Documentation License".
```

If you have no Invariant Sections, write "with no Invariant Sections" instead of saying which ones are invariant. If you have no Front-Cover Texts, write "no Front-Cover Texts" instead of "Front-Cover Texts being LIST"; likewise for Back-Cover Texts.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit

their use in free software.

Index

- ~ConSys
 - Parma_Polyhedra_Library::ConSys, [11](#)
 - ~Constraint
 - Parma_Polyhedra_Library::Constraint, [9](#)
 - ~GenSys
 - Parma_Polyhedra_Library::GenSys, [15](#)
 - ~Generator
 - Parma_Polyhedra_Library::Generator, [13](#)
 - ~LinExpression
 - Parma_Polyhedra_Library::LinExpression, [18](#)
 - ~const_iterator
 - Parma_Polyhedra_Library::ConSys::const_iterator, [12](#)
 - Parma_Polyhedra_Library::GenSys::const_iterator, [17](#)
 - add_constraints
 - Parma_Polyhedra_Library::Polyhedron, [22](#)
 - add_constraints_lazy
 - Parma_Polyhedra_Library::Polyhedron, [22](#)
 - add_dimensions_and_embed
 - Parma_Polyhedra_Library::Polyhedron, [22](#)
 - add_dimensions_and_project
 - Parma_Polyhedra_Library::Polyhedron, [22](#)
 - add_generators
 - Parma_Polyhedra_Library::Polyhedron, [22](#)
 - ALL_SATISFY
 - Parma_Polyhedra_Library, [8](#)
 - ALL_SATURATE
 - Parma_Polyhedra_Library, [8](#)
 - assign_variable
 - Parma_Polyhedra_Library::Polyhedron, [21](#)
 - begin
 - Parma_Polyhedra_Library::ConSys, [11](#)
 - Parma_Polyhedra_Library::GenSys, [15](#)
 - check_empty
 - Parma_Polyhedra_Library::Polyhedron, [23](#)
 - check_universe
 - Parma_Polyhedra_Library::Polyhedron, [23](#)
 - const_iterator
 - Parma_Polyhedra_Library::ConSys::const_iterator, [12](#)
 - Parma_Polyhedra_Library::GenSys::const_iterator, [17](#)
 - Constraint
 - Parma_Polyhedra_Library::Constraint, [9](#)
 - constraints
 - Parma_Polyhedra_Library::Polyhedron, [21](#)
 - ConSys
 - Parma_Polyhedra_Library::ConSys, [11](#)
 - convex_hull_assign
 - Parma_Polyhedra_Library::Polyhedron, [20](#)
 - convex_hull_assign_lazy
 - Parma_Polyhedra_Library::Polyhedron, [20](#)
 - Degenerate_Kind
 - Parma_Polyhedra_Library::Polyhedron, [26](#)
 - EMPTY
 - Parma_Polyhedra_Library::Polyhedron, [26](#)
 - end
 - Parma_Polyhedra_Library::ConSys, [11](#)
 - Parma_Polyhedra_Library::GenSys, [15](#)
 - Generator
 - Parma_Polyhedra_Library::Generator, [13](#)
 - generators
 - Parma_Polyhedra_Library::Polyhedron, [21](#)
 - GenSys
 - Parma_Polyhedra_Library::GenSys, [15](#)
 - GenSys_Con_Rel
 - Parma_Polyhedra_Library, [8](#)
 - id
 - Parma_Polyhedra_Library::Variable, [26](#)
 - includes
 - Parma_Polyhedra_Library::Polyhedron, [21](#)
 - insert
 - Parma_Polyhedra_Library::ConSys, [11](#)
 - Parma_Polyhedra_Library::GenSys, [15](#)
 - Parma_Polyhedra_Library::Polyhedron, [21](#)
 - intersection_assign
 - Parma_Polyhedra_Library::Polyhedron, [20](#)
 - is_empty
 - Parma_Polyhedra_Library::Polyhedron, [23](#)
 - is_equality
 - Parma_Polyhedra_Library::Constraint, [9](#)
 - is_inequality
 - Parma_Polyhedra_Library::Constraint, [9](#)
 - is_zero_dim
 - Parma_Polyhedra_Library::Polyhedron, [23](#)
 - limited_widening_assign
 - Parma_Polyhedra_Library::Polyhedron, [21](#)
 - LinExpression
 - Parma_Polyhedra_Library::LinExpression, [18](#)
 - NONE_SATISFIES
 - Parma_Polyhedra_Library, [8](#)
-

- num_dimensions
 - Parma_Polyhedra_Library::Polyhedron, 20
- OK
 - Parma_Polyhedra_Library::Polyhedron, 22
- operator *
 - Parma_Polyhedra_Library::ConSys::const_iterator, 12
 - Parma_Polyhedra_Library::GenSys::const_iterator, 17
- operator !=
 - Parma_Polyhedra_Library, 8
 - Parma_Polyhedra_Library::ConSys::const_iterator, 12
 - Parma_Polyhedra_Library::GenSys::const_iterator, 17
- operator ++
 - Parma_Polyhedra_Library::ConSys::const_iterator, 12
 - Parma_Polyhedra_Library::GenSys::const_iterator, 17
- operator ->
 - Parma_Polyhedra_Library::ConSys::const_iterator, 12
 - Parma_Polyhedra_Library::GenSys::const_iterator, 17
- operator <
 - Parma_Polyhedra_Library, 7, 8
- operator < <
 - Parma_Polyhedra_Library, 7, 8
- operator =
 - Parma_Polyhedra_Library::ConSys::const_iterator, 12
 - Parma_Polyhedra_Library::GenSys::const_iterator, 17
 - Parma_Polyhedra_Library::Polyhedron, 20
- operator ==
 - Parma_Polyhedra_Library, 8
 - Parma_Polyhedra_Library::ConSys::const_iterator, 12
 - Parma_Polyhedra_Library::GenSys::const_iterator, 17
- operator >
 - Parma_Polyhedra_Library, 8
- operator > =
 - Parma_Polyhedra_Library, 8
- Parma_Polyhedra_Library, 7
 - ALL_SATISFY, 8
 - ALL_SATURATE, 8
 - GenSys_Con_Rel, 8
 - NONE_SATISFIES, 8
 - operator <, 7, 8
 - operator < <, 7, 8
 - operator ==, 8
 - operator >, 8
 - operator > =, 8
 - SOME_SATISFY, 9
- Parma_Polyhedra_Library::Constraint, 9
 - ~Constraint, 9
 - Constraint, 9
 - is_equality, 9
 - is_inequality, 9
 - Parma_Polyhedra_Library::operator <=, 10
 - Parma_Polyhedra_Library::operator ==, 9
 - Parma_Polyhedra_Library::operator >=, 9, 10
 - Parma_Polyhedra_Library::operator > >, 10
- Parma_Polyhedra_Library::ConSys
 - ~ConSys, 11
 - begin, 11
 - ConSys, 11
 - end, 11
 - insert, 11
 - swap, 11
- Parma_Polyhedra_Library::ConSys, 10
- Parma_Polyhedra_Library::ConSys::const_iterator
 - ~const_iterator, 12
 - const_iterator, 12
 - operator *, 12
 - operator ++, 12
 - operator →, 12
 - operator =, 12
 - operator ==, 12
- Parma_Polyhedra_Library::ConSys::const_iterator, 12
- Parma_Polyhedra_Library::Generator, 13
 - ~Generator, 13
 - Generator, 13
 - Parma_Polyhedra_Library::line, 13
 - Parma_Polyhedra_Library::ray, 13
 - Parma_Polyhedra_Library::vertex, 13
 - type, 13
- Parma_Polyhedra_Library::GenSys
 - ~GenSys, 15
 - begin, 15
 - end, 15
 - GenSys, 15
 - insert, 15
 - swap, 15
- Parma_Polyhedra_Library::GenSys, 15
- Parma_Polyhedra_Library::GenSys::const_iterator
 - ~const_iterator, 17
 - const_iterator, 17
 - operator *, 17
 - operator ++, 17

- operator \rightarrow , 17
- operator=, 17
- operator==, 17
- Parma_Polyhedra_Library::GenSys::const_-
 iterator, 16
- Parma_Polyhedra_Library::line
 Parma_Polyhedra_Library::Generator, 13
- Parma_Polyhedra_Library::LinExpression
 ~LinExpression, 18
- LinExpression, 18
- Parma_Polyhedra_Library::operator *, 18
- Parma_Polyhedra_Library::operator+, 18
- Parma_Polyhedra_Library::operator+=, 19
- Parma_Polyhedra_Library::operator-, 18
- Parma_Polyhedra_Library::LinExpression, 17
- Parma_Polyhedra_Library::operator *
 Parma_Polyhedra_Library::LinExpression,
 18
- Parma_Polyhedra_Library::operator+
 Parma_Polyhedra_Library::LinExpression,
 18
- Parma_Polyhedra_Library::operator+=
 Parma_Polyhedra_Library::LinExpression,
 19
- Parma_Polyhedra_Library::operator-
 Parma_Polyhedra_Library::LinExpression,
 18
- Parma_Polyhedra_Library::operator<<
 Parma_Polyhedra_Library::Polyhedron, 23
- Parma_Polyhedra_Library::operator<=
 Parma_Polyhedra_Library::Constraint, 10
- Parma_Polyhedra_Library::Polyhedron, 23
- Parma_Polyhedra_Library::operator==
 Parma_Polyhedra_Library::Constraint, 9
- Parma_Polyhedra_Library::operator>=
 Parma_Polyhedra_Library::Constraint, 9,
 10
- Parma_Polyhedra_Library::operator>>
 Parma_Polyhedra_Library::Constraint, 10
- Parma_Polyhedra_Library::Polyhedron, 23
- Parma_Polyhedra_Library::Polyhedron, 19
- add_constraints, 22
- add_constraints_lazy, 22
- add_dimensions_and_embed, 22
- add_dimensions_and_project, 22
- add_generators, 22
- assign_variable, 21
- check_empty, 23
- check_universe, 23
- constraints, 21
- convex_hull_assign, 20
- convex_hull_assign_lazy, 20
- Degenerate_Kind, 26
- EMPTY, 26
- generators, 21
- includes, 21
- insert, 21
- intersection_assign, 20
- is_empty, 23
- is_zero_dim, 23
- limited_widening_assign, 21
- num_dimensions, 20
- OK, 22
- operator=, 20
- Parma_Polyhedra_Library::operator<<, 23
- Parma_Polyhedra_Library::operator<=, 23
- Parma_Polyhedra_Library::operator>>, 23
- Polyhedron, 20
- remove_dimensions, 22
- satisfies, 20
- substitute_variable, 21
- swap, 23
- widening_assign, 21
- ZERO_DIMENSIONAL, 26
- Parma_Polyhedra_Library::ray
 Parma_Polyhedra_Library::Generator, 13
- Parma_Polyhedra_Library::Variable, 26
- id, 26
- Variable, 26
- Parma_Polyhedra_Library::vertex
 Parma_Polyhedra_Library::Generator, 13
- Polyhedron
 Parma_Polyhedra_Library::Polyhedron, 20
- remove_dimensions
 Parma_Polyhedra_Library::Polyhedron, 22
- satisfies
 Parma_Polyhedra_Library::Polyhedron, 20
- SOME_SATISFY
 Parma_Polyhedra_Library, 9
- substitute_variable
 Parma_Polyhedra_Library::Polyhedron, 21
- swap
 Parma_Polyhedra_Library::ConSys, 11
- Parma_Polyhedra_Library::GenSys, 15
- Parma_Polyhedra_Library::Polyhedron, 23
- type
 Parma_Polyhedra_Library::Generator, 13
- Variable
 Parma_Polyhedra_Library::Variable, 26
- widening_assign
 Parma_Polyhedra_Library::Polyhedron, 21
- ZERO_DIMENSIONAL
 Parma_Polyhedra_Library::Polyhedron, 26