The Parma Polyhedra Library User's Manual[*] (version 0.4.1)

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This document describes the Parma Polyhedra Library (PPL).

For the most up-to-date information see the Parma Polyhedra Library WWW site:

http://www.cs.unipr.it/ppl/

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1 Convex Polyhedra and the PPL

1.1 A Library for Convex Polyhedra

The Parma Polyhedra Library (PPL) is a modern C++ library for the manipulation of rational convex polyhedra. Informally, a rational convex polyhedron is a set of points (in some n-dimensional vector space) that satisfies a finite number of linear inequalities having rational coefficients. The domain of convex polyhedra is employed in several systems for the analysis and verification of hardware and software components, with applications spanning imperative, functional and logic programming languages, synchronous languages and synchronization protocols, real-time and hybrid systems. Even though the PPL library is not meant to target a particular problem, the design of its interface has been largely influenced by the needs of the above class of applications. That is the reason why the library implements a few operators that are more or less specific to static analysis applications, while lacking some other operators that might be useful when working, e.g., in the field of computational geometry.

The main features of the library are the following:

- it is user friendly: you write `x + 2*y + 5*z <= 7` when you mean it;
- it is fully dynamic: available virtual memory is the only limitation to the dimension of anything;
- it provides full support for the manipulation of convex polyhedra that are not topologically closed;
- it is written in standard C++: meant to be portable;
- it is exception-safe: never leaks resources or leaves invalid object fragments around;
- it is rather efficient: and we hope to make it even more so;
- it is thoroughly documented: perhaps not literate programming but close enough;
- it is free software: distributed under the terms of the GNU General Public License.

In the following sections we describe the polyhedra and the different representations and operations supported by the PPL in more detail. For more information about the definitions and results stated here see:

- R. Bagnara, E. Ricci, E. Zaffanella and P. M. Hill - Possibly Not Closed Convex Polyhedra and the Parma Polyhedra Library - Quaderno 286 - Department of Mathematics, University of Parma, Italy, May 2002.
- K. Fukuda - Polyhedral Computation FAQ - Swiss Federal Institute of Technology, Lausanne and Zurich, Switzerland, October 2000.
- G. L. Nemhauser and L. A. Wolsey - Integer and Combinatorial Optimization - Wiley Interscience Series in Discrete Mathematics and Optimization, 1988.
- D. K. Wilde - A library for doing polyhedral operations - IRISA Publication interne n. 785, December 1993.

1.2 An Introduction to Convex Polyhedra

In this section we introduce convex polyhedra, as considered by the library, in more detail.

Vectors, Matrices and Scalar Products

We denote by $\mathbb{R}^n$ the n-dimensional vector space on the field of real numbers $\mathbb{R}$, endowed with the standard topology. The set of all non-negative reals is denoted by $\mathbb{R}_+$. For each $i \in \{0, \ldots, n-1\}$, v_i denotes the i-th component of the (column) vector $\boldsymbol{v} = (v_0, \ldots, v_{n-1})^{\mathrm{T}} \in \mathbb{R}^n$. We denote by $\mathbf{0}$ the vector of $\mathbb{R}^n$, called *the origin*, having all components equal to zero. A vector $\boldsymbol{v} \in \mathbb{R}^n$ can be also interpreted as a matrix in $\mathbb{R}^{n \times 1}$ and manipulated accordingly using the usual definitions for addition, multiplication (both by a scalar and by another matrix), and transposition, denoted by $\boldsymbol{v}^{\mathrm{T}}$.

The *scalar product* of $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, denoted $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$, is the real number

$$\boldsymbol{v}^{\mathrm{T}} \boldsymbol{w} = \sum_{i=0}^{n-1} v_i w_i.$$

For any $S_1, S_2 \subseteq \mathbb{R}^n$, the *Minkowski's sum* of S_1 and S_2 is: $S_1 + S_2 = \{\, \boldsymbol{v}_1 + \boldsymbol{v}_2 \mid \boldsymbol{v}_1 \in S_1, \boldsymbol{v}_2 \in S_2 \,\}$.

Affine Hyperplanes and Half-spaces

For each vector $\boldsymbol{a} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$, where $\boldsymbol{a} \neq \boldsymbol{0}$, and for each relational operator $\bowtie \in \{=, \geq, >\}$, the linear constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b$ defines:

- an affine hyperplane if it is an equality constraint, i.e., if $\bowtie \in \{=\}$;
- a topologically closed affine half-space if it is a non-strict inequality constraint, i.e., if $\bowtie \in \{\geq\}$;
- a topologically open affine half-space if it is a strict inequality constraint, i.e., if $\bowtie \in \{>\}$.

Note that each hyperplane $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = b$ can be defined as the intersection of the two closed affine half-spaces $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$ and $\langle -\boldsymbol{a}, \boldsymbol{x} \rangle \geq -b$. Also note that, when $\boldsymbol{a} = \boldsymbol{0}$, the constraint $\langle \boldsymbol{0}, \boldsymbol{x} \rangle \bowtie b$ is either a tautology (i.e., always true) or inconsistent (i.e., always false), so that it defines either the whole vector space $\mathbb{R}^n$ or the empty set $\varnothing$.

Convex Polyhedra

The set $\mathcal{P} \subseteq \mathbb{R}^n$ is a *not necessarily closed convex polyhedron* (*NNC polyhedron*, for short) if and only if either $\mathcal{P}$ can be expressed as the intersection of a finite number of (open or closed) affine half-spaces of $\mathbb{R}^n$ or $n = 0$ and $\mathcal{P} = \varnothing$. The set of all NNC polyhedra on the vector space $\mathbb{R}^n$ is denoted $\mathbb{P}_n$.

The set $\mathcal{P} \in \mathbb{P}_n$ is a *closed convex polyhedron* (*closed polyhedron*, for short) if and only if either $\mathcal{P}$ can be expressed as the intersection of a finite number of closed affine half-spaces of $\mathbb{R}^n$ or $n = 0$ and $\mathcal{P} = \varnothing$. The set of all closed polyhedra on the vector space $\mathbb{R}^n$ is denoted $\mathbb{CP}_n$.

When ordering NNC polyhedra by the set inclusion relation, the empty set $\varnothing$ and the vector space $\mathbb{R}^n$ are, respectively, the smallest and the biggest elements of both $\mathbb{P}_n$ and $\mathbb{CP}_n$. The vector space $\mathbb{R}^n$ is also called the *universe* polyhedron.

In theoretical terms, $\mathbb{P}_n$ is a *lattice* under set inclusion and $\mathbb{CP}_n$ is a *sub-lattice* of $\mathbb{P}_n$.

Bounded Polyhedra

An NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ is *bounded* if there exists a $\lambda \in \mathbb{R}_+$ such that

$$\mathcal{P} \subseteq \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; -\lambda \leq x_j \leq \lambda \text{ for } j = 0, \ldots, n-1 \right\}.$$

A bounded polyhedron is also called a *polytope*.

1.3 Representations of Convex Polyhedra

NNC polyhedra can be specified by using two possible representations, the constraints (or implicit) representation and the generators (or parametric) representation.

Constraints representation

In the sequel, we will simply write "equality" and "inequality" to mean "linear equality" and "linear inequality", respectively; also, we will refer to either an equality or an inequality as a *constraint*.

By definition, each polyhedron $\mathcal{P} \in \mathbb{P}_n$ is the set of solutions to a *constraint system*, i.e., a finite number of constraints. By using matrix notation, we have

$$P = \{ \boldsymbol{x} \in \mathbb{R}^n \mid A_1\boldsymbol{x} = \boldsymbol{b}_1, A_2\boldsymbol{x} \geq \boldsymbol{b}_2, A_3\boldsymbol{x} > \boldsymbol{b}_3 \},$$

where, for all $i \in \{1, 2, 3\}$, $A_i \in \mathbb{R}^{m_i} \times \mathbb{R}^n$ and $\boldsymbol{b}_i \in \mathbb{R}^{m_i}$, and $m_1, m_2, m_3 \in \mathbb{N}$ are the number of equalities, the number of non-strict inequalities, and the number of strict inequalities, respectively.

Combinations and Hulls

Let $S = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq \mathbb{R}^n$ be a finite set of vectors. For all scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, the vector $\boldsymbol{v} = \sum_{j=1}^k \lambda_j \boldsymbol{x}_j$ is said to be a *linear* combination of the vectors in S. Such a combination is said to be

- a *positive* (or *conic*) combination, if $\forall j \in \{1, \ldots, k\} : \lambda_j \in \mathbb{R}_+$;

- an *affine* combination, if $\sum_{j=1}^{k} \lambda_j = 1$;
- a *convex* combination, if it is both positive and affine.

We denote by $\mathrm{linear.hull}(S)$ (resp., $\mathrm{conic.hull}(S)$, $\mathrm{affine.hull}(S)$, $\mathrm{convex.hull}(S)$) the set of all the linear (resp., positive, affine, convex) combinations of the vectors in S.

Let $P, C \subseteq \mathbb{R}^n$, where $P \cup C = S$. We denote by $\mathrm{nnc.hull}(P, C)$ the set of all convex combinations of the vectors in S such that $\lambda_j > 0$ for some $\boldsymbol{x}_j \in P$ (informally, we say that there exists a vector of P that plays an active role in the convex combination). Note that $\mathrm{nnc.hull}(P, C) = \mathrm{nnc.hull}(P, P \cup C)$ so that, if $C \subseteq P$,

$$\mathrm{convex.hull}(P) = \mathrm{nnc.hull}(P, \varnothing) = \mathrm{nnc.hull}(P, P) = \mathrm{nnc.hull}(P, C).$$

It can be observed that $\mathrm{linear.hull}(S)$ is an affine space, $\mathrm{conic.hull}(S)$ is a topologically closed convex cone, $\mathrm{convex.hull}(S)$ is a topologically closed polytope, and $\mathrm{nnc.hull}(P, C)$ is an NNC polytope.

Points, Closure Points, Rays and Lines

Let $\mathcal{P} \in \mathbb{P}_n$ be an NNC polyhedron. Then

- a vector $\boldsymbol{p} \in \mathcal{P}$ is called a *point* of $\mathcal{P}$;
- a vector $\boldsymbol{c} \in \mathbb{R}^n$ is called a *closure point* of $\mathcal{P}$ if it is a point of the topological closure of $\mathcal{P}$;
- a vector $\boldsymbol{r} \in \mathbb{R}^n$, where $\boldsymbol{r} \neq \boldsymbol{0}$, is called a *ray* (or direction of infinity) of $\mathcal{P}$ if $\mathcal{P} \neq \varnothing$ and $\boldsymbol{p} + \lambda \boldsymbol{r} \in \mathcal{P}$, for all points $\boldsymbol{p} \in \mathcal{P}$ and all $\lambda \in \mathbb{R}_+$;
- a vector $\boldsymbol{l} \in \mathbb{R}^n$ is called a *line* of $\mathcal{P}$ if both $\boldsymbol{l}$ and $-\boldsymbol{l}$ are rays of $\mathcal{P}$.

A point of an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ is a *vertex* if and only if it cannot be expressed as a convex combination of any other pair of distinct points in $\mathcal{P}$. A ray $\boldsymbol{r}$ of a polyhedron $\mathcal{P}$ is an *extreme ray* if and only if it cannot be expressed as a positive combination of any other pair $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of rays of $\mathcal{P}$, where $\boldsymbol{r} \neq \lambda \boldsymbol{r}_1$, $\boldsymbol{r} \neq \lambda \boldsymbol{r}_2$ and $\boldsymbol{r}_1 \neq \lambda \boldsymbol{r}_2$ for all $\lambda \in \mathbb{R}_+$ (i.e., rays differing by a positive scalar factor are considered to be the same ray).

Generators Representation

Each NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ can be represented by finite sets of lines L, rays R, points P and closure points C of $\mathcal{P}$. The 4-tuple $\mathcal{G} = (L, R, P, C)$ is said to be a *generator system* for $\mathcal{P}$, in the sense that

$$\mathcal{P} = \mathrm{linear.hull}(L) + \mathrm{conic.hull}(R) + \mathrm{nnc.hull}(P, C),$$

where the symbol '$+$' denotes the Minkowski's sum.

When $\mathcal{P} \in \mathbb{CP}_n$ is a closed polyhedron, then it can be represented by finite sets of lines L, rays R and points P of $\mathcal{P}$. In this case, the 3-tuple $\mathcal{G} = (L, R, P)$ is said to be a *generator system* for $\mathcal{P}$ since we have

$$\mathcal{P} = \mathrm{linear.hull}(L) + \mathrm{conic.hull}(R) + \mathrm{convex.hull}(P).$$

Thus, in this case, every closure point of $\mathcal{P}$ is a point of $\mathcal{P}$.

For any $\mathcal{P} \in \mathbb{P}_n$ and generator system $\mathcal{G} = (L, R, P, C)$ for $\mathcal{P}$, we have $\mathcal{P} = \varnothing$ if and only if $P = \varnothing$. Also P must contain all the vertices of $\mathcal{P}$ although $\mathcal{P}$ can be non-empty and have no vertices. In this case, as P is necessarily non-empty, it must contain points of $\mathcal{P}$ that are *not* vertices. For instance, the half-space of $\mathbb{R}^2$ corresponding to the single constraint $y \geq 0$ can be represented by the generator system $\mathcal{G} = (L, R, P)$ such that $L = \{(1, 0)^{\mathrm{T}}\}$, $R = \{(0, 1)^{\mathrm{T}}\}$, and $P = \{(0, 0)^{\mathrm{T}}\}$. It is also worth noting that the only ray in R is *not* an extreme ray of $\mathcal{P}$.

Minimized Representations

A constraints system $\mathcal{C}$ for an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ is said to be *minimized* if no proper subset of $\mathcal{C}$ is a constraint system for $\mathcal{P}$.

Similarly, a generator system $\mathcal{G} = (L, R, P, C)$ for an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ is said to be *minimized* if there does not exist a generator system $\mathcal{G}' = (L', R', P', C') \neq \mathcal{G}$ for $\mathcal{P}$ such that $L' \subseteq L$, $R' \subseteq R$, $P' \subseteq P$ and $C' \subseteq C$.

Double Description

Any NNC polyhedron $\mathcal{P}$ can be described by using a constraint system $\mathcal{C}$, a generator system $\mathcal{C}$, or both by means of the *double description pair (DD pair)* $(\mathcal{C}, \mathcal{G})$. The *double description method* is a collection of well-known as well as novel theoretical results showing that, given one kind of representation, there are algorithms for computing a representation of the other kind and for minimizing both representations by removing redundant constraints/generators.

Such changes of representation form a key step in the implementation of many operators on NNC polyhedra: this is because some operators, such as intersections and poly-hulls, are provided with a natural and efficient implementation when using one of the representations in a DD pair, while being rather cumbersome when using the other.

Topologies and Topological-compatibility

As indicated above, when an NNC polyhedron $\mathcal{P}$ is necessarily closed, we can ignore the closure points contained in its generator system $\mathcal{G} = (L, R, P, C)$ (as every closure point is also a point) and represent $\mathcal{P}$ by the triple (L, R, P). Similarly, $\mathcal{P}$ can be represented by a constraint system that has no strict inequalities. Thus a necessarily closed polyhedron can have a smaller representation than one that is not necessarily closed. Moreover, operators restricted to work on closed polyhedra only can be implemented more efficiently. For this reason the library provides two alternative "topological kinds" for a polyhedron, *NNC* and *C*. We shall abuse terminology by referring to the topologcal kind of a polyhedron as its *topology*.

In the library, the topology of each polyhedron object is fixed once for all at the time of its creation and must be respected when performing operations on the polyhedron.

Unless it is otherwise stated, all the polyhedra, constraints and/or generators in any library operation must obey the following *topological-compatibility* rules:

- polyhedra are tologically-compatible if and only if they have the same topology;
- all constraints except for strict constraints and all generators except for closure points are topologically-compatible with both C and NNC polyhedra;
- strict inequality constraints and closure points are topologically-compatible with a polyhedron if and only if it is NNC.

Wherever possible, the library provides methods that, starting from a polyhedron of a given topology, build the corresponding polyhedron having the other topology.

Space Dimensions and Dimension-compatibility

The *space dimension* of an NNC polyhedron $P \in \mathbb{P}_n$ (resp., a C polyhedron $P \in \mathbb{CP}_n$) is the dimension $n \in \mathbb{N}$ of the corresponding vector space $\mathbb{R}^n$. The space dimension of constraints, generators and other objects of the library is defined similarly.

Unless it is otherwise stated, all the polyhedra, constraints and/or generators in any library operation must obey the following space *dimension-compatibility* rules:

- polyhedra are dimension-compatible if and only if they have the same space dimension;
- the constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b$ where $\bowtie \in \{=, \geq, >\}$ and $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^m$, is dimension-compatible with a polyhedron having space dimension n if and only if $m \leq n$;
- the generator $\boldsymbol{x} \in \mathbb{R}^m$ is dimension-compatible with a polyhedron having space dimension n if and only if $m \leq n$;
- a system of constraints (resp., generators) is dimension-compatible with a polyhedron if and only if all the constraints (resp., generators) in the system are dimension-compatible with the polyhedron.

While the space dimension of a constraint, a generator or a system thereof is automatically adjusted when needed, the space dimensions of polyhedra can only be changed by explicit calls to operators provided for that purpose.

Rational Polyhedra

An NNC polyhedron is called *rational* if it can be represented by a constraint system where all the constraints have rational coefficients. It has been shown that an NNC polyhedron is rational if and only if it can be represented by a generator system where all the generators have rational coefficients.

The library only supports rational polyhedra. The restriction to rational numbers applies not only to polyhedra, but also to the other numeric arguments that may be required by the operators considered, such as the coefficients defining (rational) affine transformations and (rational) bounding boxes.

1.4 Operations on Convex Polyhedra

In this section we briefly describe operations on NNC polyhedra that are provided by the library.

Intersection and Convex Polyhedral Hull

For any pair of NNC polyhedra $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{P}_n$, the *intersection* of $\mathcal{P}_1$ and $\mathcal{P}_2$, defined as the set intersection $\mathcal{P}_1 \cap \mathcal{P}_2$, is the biggest NNC polyhedron included in both $\mathcal{P}_1$ and $\mathcal{P}_2$; similarly, the *convex polyhedral hull* (or *poly-hull*) of $\mathcal{P}_1$ and $\mathcal{P}_2$, denoted by $\mathcal{P}_1 \uplus \mathcal{P}_2$, is the smallest NNC polyhedron that includes both $\mathcal{P}_1$ and $\mathcal{P}_2$. The intersection and poly-hull of any pair of closed polyhedra in $\mathbb{CP}_n$ is also closed.

In theoretical terms, the intersection and poly-hull operators defined above are the binary *meet* and the binary *join* operators on the lattices $\mathbb{P}_n$ and $\mathbb{CP}_n$.

Convex Polyhedral Difference

For any pair of NNC polyhedra $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{P}_n$, the *convex polyhedral difference* (or *poly-difference*) of $\mathcal{P}_1$ and $\mathcal{P}_2$ is defined as the poly-hull of the set-theoretic difference of $\mathcal{P}_1$ and $\mathcal{P}_2$.

In general, even though $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{CP}_n$ are topologically closed polyhedra, their poly-difference may be a convex polyhedron that is not topologically closed. For this reason, when computing the poly-difference of two C polyhedra, the library will enforce the topological closure of the result.

Adding New Dimensions to the Vector Space

The library provides two operators for increasing the space dimension of an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$, therefore transforming it into a new NNC polyhedron $\mathcal{Q} \in \mathbb{P}_m$, where $m > n$. In both cases, the added dimensions of the vector space are those having the highest indices.

The operator *embedding* the polyhedron $\mathcal{P}$ into the new vector space will return the polyhedron $\mathcal{Q}$ defined by all and only the constraints defining $\mathcal{P}$ (the variables corresponding to the added dimensions are unconstrained). For instance, when starting from a polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and adding a third dimension, the result will be the polyhedron

$$\mathcal{Q} = \bigl\{ (x_0, x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^3 \bigm| (x_0, x_1)^{\mathrm{T}} \in \mathcal{P} \bigr\}.$$

In contrast, the operator *projecting* the polyhedron $\mathcal{P}$ into the new vector space will return the polyhedron $\mathcal{Q}$ whose constraint system, besides the constraints defining $\mathcal{P}$, will include additional constraints on the added dimensions. Namely, the corresponding variables are all constrained to be equal to 0. For instance, when starting from a polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and adding a third dimension, the result will be the polyhedron

$$\mathcal{Q} = \bigl\{ (x_0, x_1, 0)^{\mathrm{T}} \in \mathbb{R}^3 \bigm| (x_0, x_1)^{\mathrm{T}} \in \mathcal{P} \bigr\}.$$

Removing Dimensions from the Vector Space

The library provides two operators for decreasing the space dimension of an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$, therefore transforming it into a new NNC polyhedron $\mathcal{Q} \in \mathbb{P}_m$, where $m < n$.

Given a set of variables, there is an operator that will remove all the space dimensions corresponding to the variables in this set. For instance, letting $\mathcal{P} \in \mathbb{P}_4$ be the singleton set $\bigl\{(3,1,0,2)^{\mathrm{T}}\bigr\} \subseteq \mathbb{R}^4$, then after invoking this operator with the set of variables $\{x_1, x_2\}$ the resulting polyhedron is

$$\mathcal{Q} = \bigl\{(3,2)^{\mathrm{T}}\bigr\} \subseteq \mathbb{R}^2.$$

Another operator removes from the vector space all the dimensions having an index greater than or equal to m. For instance, letting $\mathcal{P} \in \mathbb{P}_4$ defined as before, by invoking this operator with $m = 2$ the resulting polyhedron will be

$$\mathcal{Q} = \bigl\{(3,1)^{\mathrm{T}}\bigr\} \subseteq \mathbb{R}^2.$$

Affine Images and Preimages

The function mapping $\phi \colon \mathbb{R}^n \to \mathbb{R}^m$ is an *affine transformation* if there exist a matrix $A \in \mathbb{R}^m \times \mathbb{R}^n$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$ such that, for all $\boldsymbol{x} \in \mathbb{R}^n$, we have $\phi(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$. If $n = m$, then the function ϕ is said to be *space-dimension preserving*. We denote by $\phi(S) \subseteq \mathbb{R}^m$ the *image* under ϕ of the set $S \subseteq \mathbb{R}^n$; similarly, we denote by $\phi^{-1}(S') \subseteq \mathbb{R}^n$ the *preimage* under ϕ of $S' \subseteq \mathbb{R}^m$, that is the largest set $S \subseteq \mathbb{R}^n$ such that $\phi(S) = S'$.

Both $\mathbb{P}_n$ and $\mathbb{CP}_n$ are closed under the application of any space-dimension preserving affine image and preimage operators.

The library provides two operators, one computes an affine image and the other an affine preimage of a polyhedron $\mathcal{P} \in \mathbb{P}_n$ for a given variable x_k and linear expression E

$$\sum_{i=0}^{n-1} a_i x_i + b.$$

This variable and expression determine the affine transformation ϕ that is to be used by the operator. That is, ϕ is the transformation defined by the matrix and vector

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_0 & a_1 & \dots & a_{k-1} & a_k & a_{k+1} & \dots & a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b \\ \vdots \\ 0 \end{pmatrix}$$

where the a_i (resp., b) occurs in the $(k+1)$st row in A (resp., position in $\boldsymbol{b}$). Thus ϕ transforms any point $(x_0, \dots, x_{n-1})^{\mathrm{T}}$ in the polyhedron $\mathcal{P}$ to

$$\Bigl(x_0, \dots, \bigl(\textstyle\sum_{i=0}^{n-1} a_i x_i + b\bigr), \dots, x_{n-1}\Bigr)^{\mathrm{T}}.$$

The affine image operator computes the affine image of $\mathcal{P}$ under ϕ. For instance, suppose the polyhedron $\mathcal{P}$ to be transformed is the square in $\mathbb{R}^2$ generated by the set of points $\bigl\{(0,0),(0,3),(3,0),(3,3)\bigr\}$. Then, for example if the considered variable is x_0 and the linear expression $x_0 + 2x_1 + 4$ (so that $k = 0$, $a_0 = 1, a_1 = 2, b = 4$), the affine image operator will translate $\mathcal{P}$ to the parallelogram $\mathcal{P}_1$ generated by the set of points $\bigl\{(4,0),(10,3),(7,0),(13,3)\bigr\}$ with height equal to the side of the square and oblique sides parallel to the line $x_0 - 2x_1$. If the considered variable is as before (i.e., $k = 0$) but the linear expression is x_1 (so that $a_0 = 0, a_1 = 1, b = 0$), then the resulting polyhedron $\mathcal{P}_2$ is the positive diagonal of the square.

The affine preimage operator computes the affine preimage of $\mathcal{P}$ under ϕ. For instance, suppose now that we apply the affine preimage operator as given in the first example using variable x_0 and linear expression $x_0 + 2x_1 + 4$ to the parallelogram $\mathcal{P}_1$; then we get the original square $\mathcal{P}$ back. If, on the other hand, we apply the affine preimage operator as given in the second example using variable x_0 and linear expression x_1 to $\mathcal{P}_2$, then the resulting polyhedron is a line that corresponds to the x_1 axes.

Observe that provided the coefficient a_k of the considered variable in the linear expression is non-zero, the affine transformation is invertible.

Relation-with Operators

The library provides operators for checking the relation holding between an NNC polyhedron and either a constraint or a generator.

Suppose $\mathcal{P}$ is an NNC polyhedron and $\mathcal{C}$ an arbitrary constraint system representing $\mathcal{P}$. Suppose also that $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr)$ is a constraint with $\bowtie \in \{=, \geq, >\}$ and $\mathcal{Q}$ the set of points that satisfy c. The possible relations between $\mathcal{P}$ and c are as follows.

- $\mathcal{P}$ *is disjoint* from c if $\mathcal{P} \cap \mathcal{Q} = \varnothing$; that is, adding c to $\mathcal{C}$ gives us the empty polyhedron.
- $\mathcal{P}$ *strictly intersects* c if $\mathcal{P} \cap \mathcal{Q} \neq \varnothing$ and $\mathcal{P} \cap \mathcal{Q} \subset \mathcal{P}$; that is, adding c to $\mathcal{C}$ gives us a non-empty polyhedron strictly smaller than $\mathcal{P}$.
- $\mathcal{P}$ *is included* in c if $\mathcal{P} \subseteq \mathcal{Q}$; that is, adding c to $\mathcal{C}$ leaves $\mathcal{P}$ unchanged.
- $\mathcal{P}$ *saturates* c if $\mathcal{P} \subseteq \mathcal{H}$, where $\mathcal{H}$ is the hyperplane induced by constraint c, i.e., the set of points satisfying the equality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = b$; that is, adding the constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = b$ to $\mathcal{C}$ leaves $\mathcal{P}$ unchanged.

The polyhedron $\mathcal{P}$ *subsumes* the generator g if adding g to any generator system representing $\mathcal{P}$ does not change $\mathcal{P}$.

Widening Operators

The library provides widening operators for the domain of NNC polyhedra. These operators use a widening, we call *H79-widening*, which is based on that introduced in N. Halbwachs, *Détermination automatique de relations linéaires vérifiées par les variables d'un programme*, Thèse de 3ème cicle d'informatique, Université scientifique et médicale de Grenoble, Grenoble, France, March 1979. This widening is also described in N. Halbwachs, Y.-E. Proy, and P. Roumanoff, Verification of real-time systems using linear relation analysis, *Formal Methods in System Design*, 11(2):157-185, 1997.

There are a few differences between the H79-widening and the widenings described in the cited paper. In particular, the H79-widening of an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ using the NNC polyhedron $\mathcal{Q} \in \mathbb{P}_n$:

- allows for equalities in $\mathcal{P}$ and $\mathcal{Q}$ (the original definition is restricted to inequalities);
- does not require $\mathcal{P}$ and $\mathcal{Q}$ to be topologically closed;
- requires as a precondition that $\mathcal{Q} \subseteq \mathcal{P}$.

Time-Elapse Operator

The *time-elapse* operator has been defined in N. Halbwachs and Y.-E. Proy and P. Roumanoff, Verification of Real-Time Systems using Linear Relation Analysis, in *Formal Methods in System Design* 11(2):157–185, 1997.

Actually, the time-elapse operator provided by the library is a slight generalization of that one, since it also works on NNC polyhedra. For any two NNC polyhedra $\mathcal{P}, \mathcal{Q} \in \mathbb{P}_n$, the time-elapse between $\mathcal{P}$ and $\mathcal{Q}$, denoted $\mathcal{P} \nearrow \mathcal{Q}$, is the smallest NNC polyhedron containing the set

$$\bigl\{\, \boldsymbol{p} + \lambda \boldsymbol{q} \in \mathbb{R}^n \bigm| \boldsymbol{p} \in \mathcal{P}, \boldsymbol{q} \in \mathcal{Q}, \lambda \in \mathbb{R}_+ \,\bigr\}.$$

Note that, if $\mathcal{P}, \mathcal{Q} \in \mathbb{CP}_n$ are closed polyhedra, the above set is also a closed polyhedron. In contrast, when $\mathcal{Q}$ is not topologically closed, the above set might not be an NNC polyhedron.

Intervals, boxes and bounding boxes

An *interval* in $\mathbb{R}$ is a pair of *bounds*, called *lower* and *upper*. Each bound can be either (1) *closed and bounded*, (2) *open and bounded*, or (3) *open and unbounded*. If the bound is *bounded*, then it has a value in $\mathbb{R}$. An n-dimensional *box* $\mathcal{B}$ in $\mathbb{R}^n$ is a sequence of n intervals in $\mathbb{R}$.

The polyhedron $\mathcal{P}$ *represents a box* $\mathcal{B}$ in $\mathbb{R}^n$ if $\mathcal{P}$ is described by a constraint system in $\mathbb{R}^n$ that consists of one constraint for each bounded bound (lower and upper) in an interval in $\mathcal{B}$: Letting $\boldsymbol{e}_i = (0, \ldots, 1, \ldots, 0)$ be the vector in $\mathbb{R}^n$ with 1 in the i'th position and zeros in every other position; if the lower bound of the i'th interval in $\mathcal{B}$ is bounded, the corresponding constraint is defined as $\langle \boldsymbol{e}_i, \boldsymbol{x} \rangle \bowtie b$, where b is the value of the bound and $\bowtie$ is $\geq$ if it is a closed bound and $>$ if it is an open bound. Similarly, if the upper bound of the i'th interval in $\mathcal{B}$ is bounded, the corresponding constraint is defined as $\langle \boldsymbol{e}_i, \boldsymbol{x} \rangle \bowtie b$, where b is the value of the bound and $\bowtie$ is $\leq$ if it is a closed bound and $<$ if it is an open bound.

If every bound in the intervals defining a box $\mathcal{B}$ is either closed and bounded or open and unbounded, then $\mathcal{B}$ represents a closed polyhedron.

The *bounding box* of an NNC polyhedron $\mathcal{P}$ is the smallest n-dimensional box containing $\mathcal{P}$.

The library provides operations for computing the bounding box of an NNC polyhedron and conversely, for obtaining the NNC polyhedron representing a given bounding box.

2 PPL Namespace Index

2.1 PPL Namespace List

Here is a list of all documented namespaces with brief descriptions:

3 PPL Hierarchical Index

3.1 PPL Class Hierarchy

This inheritance list is sorted roughly, but not completely, alphabetically:

4 PPL Compound Index

4.1 PPL Compound List

Here are the classes, structs, unions and interfaces with brief descriptions:

5 PPL File Index

5.1 PPL File List

Here is a list of all documented files with brief descriptions:

6 PPL Page Index

6.1 PPL Related Pages

Here is a list of all related documentation pages:

7 PPL Namespace Documentation

7.1 Parma_Polyhedra_Library Namespace Reference

The entire library is confined into this namespace.

Compounds

- class Variable

  A dimension of the space.

- class LinExpression

  A linear expression.

- class Constraint

  A linear equality or inequality.

- class Generator

  A line, ray, point or closure point.

- class Poly_Con_Relation

  The relation between a polyhedron and a constraint.

- class Poly_Gen_Relation

  The relation between a polyhedron and a generator.

- class Polyhedron

  The base class for convex polyhedra.

- class C_Polyhedron

  A closed convex polyhedron.

- class NNC_Polyhedron

  A not necessarily closed convex polyhedron.

- class Throwable

  User objects' the PPL can throw.

- struct From_Bounding_Box

  A tag class.

Typedefs

- typedef mpz_class Integer

  See the GMP's manual available at `http://swox.com/gmp/` .

Functions

- Generator line (const LinExpression &e)

  Shorthand for Generator Generator line(const LinExpression& e).

- Generator ray (const LinExpression &e)

  Shorthand for Generator Generator::ray(const LinExpression& e).

- Generator point (const LinExpression &e=LinExpression::zero(), const Integer &d=Integer_one())

  Shorthand for Generator point(const LinExpression& e, const Integer& d).

- Generator closure_point (const LinExpression &e=LinExpression::zero(), const Integer &d=Integer_one())

  Shorthand for Generator Generator::closure_point(const LinExpression& e, const Integer& d).

Variables

- const Throwable ∗volatile abandon_exponential_computations

7.1.1 Detailed Description

The entire library is confined into this namespace.

7.1.2 Variable Documentation

7.1.2.1 const Throwable∗ volatile Parma_Polyhedra_Library::abandon_exponential_computations

This pointer, which is initialized to zero, is repeatedly checked along any exponential computation path in the library. When it is found nonzero the exception it points to is thrown. In other words, making this pointer point to an exception (and leaving it in this state) ensures that the library will return control to the client application, possibly by throwing the given exception, within a time that is a linear function of the space dimension of the object (polyhedron, system of constraints or generators) of highest dimension on which the library is operating upon.

Note:

The only sensible way to assign to this pointer is from within a signal handler or from a parallel thread. For this reason, the library, apart from ensuring that the pointer is initially set to zero, never assigns to it. In particular, it does not zero it again when the exception is thrown: it is the client's responsibility to do so.

7.2 std Namespace Reference

The standard C++ namespace.

7.2.1 Detailed Description

The standard C++ namespace.

The Parma Polyhedra Library conforms to the C++ standard and, in particular, as far as reserved names are concerned (17.4.3.1, [lib.reserved.names]). The PPL, however, defines several template specializations for the standard library templates swap() and iter_swap() (25.2.2, [lib.alg.swap]).

8 PPL Class Documentation

8.1 Parma_Polyhedra_Library::C_Polyhedron Class Reference

A closed convex polyhedron.

Inherits Parma_Polyhedra_Library::Polyhedron.

Public Methods

- C_Polyhedron (size_t num_dimensions=0, Degenerate_Kind kind=UNIVERSE)

  Builds either the universe or the empty C polyhedron.

- C_Polyhedron (ConSys &cs)

  Builds a C polyhedron from a system of constraints.

- C_Polyhedron (GenSys &gs)

  Builds a C polyhedron from a system of generators.

- C_Polyhedron (const NNC_Polyhedron &y)

  Builds a C polyhedron from the NNC_Polyhedron `y`.

- template<class Box> C_Polyhedron (const Box &box, From_Bounding_Box dummy)

  Builds a C polyhedron out of a generic, interval-based bounding box.

- C_Polyhedron (const C_Polyhedron &y)

  Ordinary copy-constructor.

- C_Polyhedron & operator= (const C_Polyhedron &y)

  The assignment operator. (`*this` *and* `y` *can be dimension-incompatible.).*

- ~C_Polyhedron ()

  Destructor.

8.1.1 Detailed Description

A closed convex polyhedron.

An object of the class C_Polyhedron represents a *topologically closed* convex polyhedron in the vector space $\mathbb{R}^n$.

When building a closed polyhedron starting from a system of constraints, an exception is thrown if the system contains a *strict inequality* constraint. Similarly, an exception is thrown when building a closed polyhedron starting from a system of generators containing a *closure point*.

Note:

Such an exception will be obtained even if the system of constraints (resp., generators) actually defines a topologically closed subset of the vector space, i.e., even if all the strict inequalities (resp., closure points) in the system happen to be redundant with respect to the system obtained by removing all the strict inequality constraints (resp., all the closure points). In contrast, when building a closed polyhedron starting from an object of the class NNC_Polyhedron, the precise topological closure test will be performed.

8.1.2 Constructor & Destructor Documentation

8.1.2.1 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (size_t *num_dimensions* = 0, Degenerate_Kind *kind* = UNIVERSE) `[explicit]`

Builds either the universe or the empty C polyhedron.

Parameters:

num_dimensions The number of dimensions of the vector space enclosing the C polyhedron.

kind Specifies whether a universe or an empty C polyhedron should be built.

Both parameters are optional: by default, a 0-dimension space universe C polyhedron is built.

8.1.2.2 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (ConSys & *cs*)

Builds a C polyhedron from a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

cs The system of constraints defining the polyhedron. It is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if the system of constraints contains strict inequalities.

8.1.2.3 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (GenSys & *gs*)

Builds a C polyhedron from a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

gs The system of generators defining the polyhedron. It is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if the system of generators is not empty but has no points, or if it contains closure points.

8.1.2.4 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (const NNC_Polyhedron & *y*) `[explicit]`

Builds a C polyhedron from the NNC_Polyhedron `y`.

Exceptions:

std::invalid_argument thrown if the polyhedron `y` is not topologically closed.

8.1.2.5 template<class Box> Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (const Box & *box*, From_Bounding_Box *dummy*)

Builds a C polyhedron out of a generic, interval-based bounding box.

For a description of the methods that should be provided by the template class Box, see the documentation of the protected method: template <class Box> Polyhedron::Polyhedron(Topology topol, const Box& box);

Parameters:

box The bounding box representing the polyhedron to be built.

dummy A dummy tag to syntactically differentiate this one from the other constructors.

Exceptions:

std::invalid_argument thrown if `box` has intervals that are not topologically closed (i.e., having some finite but open bounds).

8.2 Parma_Polyhedra_Library::Constraint Class Reference

A linear equality or inequality.

Public Types

- enum Type { EQUALITY, NONSTRICT_INEQUALITY, STRICT_INEQUALITY }

  The constraint type.

Public Methods

- Constraint (const Constraint &c)

  Ordinary copy-constructor.

- ~Constraint ()

  Destructor.

- Constraint & operator= (const Constraint &c)

  Assignment operator.

- size_t space_dimension () const

  Returns the dimension of the vector space enclosing ∗`this`.

- Type type () const

  Returns the constraint type of ∗`this`.

- bool is_equality () const

  Returns `true` *if and only if* ∗`this` *is an equality constraint.*

- bool is_inequality () const

  Returns `true` *if and only if* ∗`this` *is an inequality constraint (either strict or non-strict).*

- bool is_nonstrict_inequality () const

Returns `true` *if and only if* `*this` *is a non-strict inequality constraint.*

- bool is_strict_inequality () const

  Returns `true` *if and only if* `*this` *is a strict inequality constraint.*

- const Integer & coefficient (Variable v) const

  Returns the coefficient of `v` *in* `*this`.

- const Integer & inhomogeneous_term () const

  Returns the inhomogeneous term of `*this`.

- bool OK () const

  Checks if all the invariants are satisfied.

Static Public Methods

- const Constraint & zero_dim_false ()

  The unsatisfiable (zero-dimension space) constraint $0 = 1$.

- const Constraint & zero_dim_positivity ()

  The true (zero-dimension space) constraint $0 \leq 1$*, also known as* positivity constraint.

Friends

- class Parma_Polyhedra_Library::Polyhedron
- Constraint Parma_Polyhedra_Library::operator== (const LinExpression &e1, const LinExpression &e2)

  Returns the constraint `e1` = `e2`.

- Constraint Parma_Polyhedra_Library::operator== (const LinExpression &e, const Integer &n)

  Returns the constraint `e` = `n`.

- Constraint Parma_Polyhedra_Library::operator== (const Integer &n, const LinExpression &e)

  Returns the constraint `n` = `e`.

- Constraint Parma_Polyhedra_Library::operator>= (const LinExpression &e1, const LinExpression &e2)

  Returns the constraint `e1` >= `e2`.

- Constraint Parma_Polyhedra_Library::operator>= (const LinExpression &e, const Integer &n)

  Returns the constraint `e` >= `n`.

- Constraint Parma_Polyhedra_Library::operator>= (const Integer &n, const LinExpression &e)

  Returns the constraint `n` >= `e`.

- Constraint Parma_Polyhedra_Library::operator<= (const LinExpression &e1, const LinExpression &e2)

  Returns the constraint `e1` <= `e2`.

- Constraint Parma_Polyhedra_Library::operator<= (const LinExpression &e, const Integer &n)
  Returns the constraint `e` $<=$ `n`.
- Constraint Parma_Polyhedra_Library::operator<= (const Integer &n, const LinExpression &e)
  Returns the constraint `n` $<=$ `e`.
- Constraint Parma_Polyhedra_Library::operator> (const LinExpression &e1, const LinExpression &e2)
  Returns the constraint `e1` $>$ `e2`.
- Constraint Parma_Polyhedra_Library::operator> (const LinExpression &e, const Integer &n)
  Returns the constraint `e` $>$ `n`.
- Constraint Parma_Polyhedra_Library::operator> (const Integer &n, const LinExpression &e)
  Returns the constraint `n` $>$ `e`.
- Constraint Parma_Polyhedra_Library::operator< (const LinExpression &e1, const LinExpression &e2)
  Returns the constraint `e1` $<$ `e2`.
- Constraint Parma_Polyhedra_Library::operator< (const LinExpression &e, const Integer &n)
  Returns the constraint `e` $<$ `n`.
- Constraint Parma_Polyhedra_Library::operator< (const Integer &n, const LinExpression &e)
  Returns the constraint `n` $<$ `e`.
- Constraint Parma_Polyhedra_Library::operator>> (const Constraint &c, unsigned int offset)
  Returns the constraint `c` *with variables renamed by adding* `offset` *to their Cartesian axis identifier.*

Related Functions

(Note that these are not member functions.)

- std::ostream & operator<< (std::ostream &s, const Constraint &c)
  Output operator.
- void swap (Parma_Polyhedra_Library::Constraint &x, Parma_Polyhedra_Library::Constraint &y)
  Specializes `std::swap`.

8.2.1 Detailed Description

A linear equality or inequality.

An object of the class Constraint is either:

- an equality: $\sum_{i=0}^{n-1} a_i x_i + b = 0$;
- a non-strict inequality: $\sum_{i=0}^{n-1} a_i x_i + b \geq 0$; or

- a strict inequality: $\sum_{i=0}^{n-1} a_i x_i + b > 0$;

where n is the dimension of the space, a_i is the integer coefficient of variable x_i and b is the integer inhomogeneous term.

How to build a constraint

Constraints are typically built by applying a relational operator to a pair of linear expressions. Available relational operators include equality (==), non-strict inequalities (>= and <=) and strict inequalities (< and >). The space-dimension of a constraint is defined as the maximum space-dimension of the arguments of its constructor.
In the following examples it is assumed that variables x, y and z are defined as follows:

```
Variable x(0);
Variable y(1);
Variable z(2);
```

Example 1

The following code builds the equality constraint $3x + 5y - z = 0$, having space-dimension 3:

```
Constraint eq_c(3*x + 5*y - z == 0);
```

The following code builds the (non-strict) inequality constraint $4x \geq 2y - 13$, having space-dimension 2:

```
Constraint ineq_c(4*x >= 2*y - 13);
```

The corresponding strict inequality constraint $4x > 2y - 13$ is obtained as follows:

```
Constraint strict_ineq_c(4*x > 2*y - 13);
```

An unsatisfiable constraint on the zero-dimension space $\mathbb{R}^0$ can be specified as follows:

```
Constraint false_c = Constraint::zero_dim_false();
```

Equivalent, but more involved ways are the following:

```
Constraint false_c1(LinExpression::zero() == 1);
Constraint false_c2(LinExpression::zero() >= 1);
Constraint false_c3(LinExpression::zero() > 0);
```

In constrast, the following code defines an unsatisfiable constraint having space-dimension 3:

```
Constraint false_c(0*z == 1);
```

How to inspect a constraint

Several methods are provided to examine a constraint and extract all the encoded information: its space-dimension, its type (equality, non-strict inequality, strict inequality) and the value of its integer coefficients.

Example 2

The following code shows how it is possible to access each single coefficient of a constraint. Given an inequality constraint (in this case $x - 5y + 3z <= 4$), we construct a new constraint corresponding to its complement (thus, in this case we want to obtain the strict inequality constraint $x - 5y + 3z > 4$).

```
Constraint c1(x - 5*y + 3*z <= 4);
cout << "Constraint c1: " << c1 << endl;
if (c1.is_equality())
  cout << "Constraint c1 is not an inequality." << endl;
else {
  LinExpression e;
```

```
    for (int i = c1.space_dimension() - 1; i >= 0; i--)
      e += c1.coefficient(Variable(i)) * Variable(i);
    e += c1.inhomogeneous_term();
    Constraint c2 = c1.is_strict_inequality() ? (e <= 0) : (e < 0);
    cout << "Complement c2: " << c2 << endl;
  }
```

The actual output is the following:

```
Constraint c1: -A + 5*B - 3*C >= -4
Complement c2: A - 5*B + 3*C > 4
```

Note that, in general, the particular output obtained can be syntactically different from the (semantically equivalent) constraint considered.

8.2.2 Member Enumeration Documentation

8.2.2.1 enum Parma_Polyhedra_Library::Constraint::Type

The constraint type.

Enumeration values:

EQUALITY The constraint is an equality.

NONSTRICT_INEQUALITY The constraint is a non-strict inequality.

STRICT_INEQUALITY The constraint is a strict inequality.

8.2.3 Member Function Documentation

8.2.3.1 const Integer& Parma_Polyhedra_Library::Constraint::coefficient (Variable *v*) const

Returns the coefficient of `v` in `*this`.

Exceptions:

std::invalid_argument thrown if the index of `v` is greater than or equal to the space-dimension of `*this`.

8.3 Parma_Polyhedra_Library::From_Bounding_Box Struct Reference

A tag class.

8.3.1 Detailed Description

A tag class.

Tag class to differentiate the C_Polyhedron and NNC_Polyhedron constructors that build a polyhedron out of a bounding box.

8.4 Parma_Polyhedra_Library::Generator Class Reference

A line, ray, point or closure point.

Public Types

- enum Type { LINE, RAY, POINT, CLOSURE_POINT }

  The generator type.

Public Methods

- Generator (const Generator &g)

  Ordinary copy-constructor.

- ~Generator ()

  Destructor.

- Generator & operator= (const Generator &g)

  Assignment operator.

- size_t space_dimension () const

  Returns the dimension of the vector space enclosing ∗`this`.

- Type type () const

  Returns the generator type of ∗`this`.

- bool is_line () const

  Returns `true` *if and only if* ∗`this` *is a line.*

- bool is_ray () const

  Returns `true` *if and only if* ∗`this` *is a ray.*

- bool is_point () const

  Returns `true` *if and only if* ∗`this` *is a point.*

- bool is_closure_point () const

  Returns `true` *if and only if* ∗`this` *is a closure point.*

- const Integer & coefficient (Variable v) const

  Returns the coefficient of `v` *in* ∗`this`.

- const Integer & divisor () const

  If ∗`this` *is either a point or a closure point, returns its divisor.*

- bool OK () const

  Checks if all the invariants are satisfied.

Static Public Methods

- Generator line (const LinExpression &e)

  Returns the line of direction e.

- Generator ray (const LinExpression &e)

  Returns the ray of direction e.

- Generator point (const LinExpression &e=LinExpression::zero(), const Integer &d=Integer_one())

  Returns the point at e / d.

- Generator closure_point (const LinExpression &e=LinExpression::zero(), const Integer &d=Integer_one())

  Returns the closure point at e / d.

- const Generator & zero_dim_point ()

  Returns the origin of the zero-dimensional space $\mathbb{R}^0$.

- const Generator & zero_dim_closure_point ()

  Returns, as a closure point, the origin of the zero-dimensional space $\mathbb{R}^0$.

Friends

- class Parma_Polyhedra_Library::Polyhedron
- std::ostream & Parma_Polyhedra_Library::operator<< (std::ostream &s, const Generator &g)

  Output operator.

Related Functions

(Note that these are not member functions.)

- void swap (Parma_Polyhedra_Library::Generator &x, Parma_Polyhedra_Library::Generator &y)

  Specializes `std::swap`.

8.4.1 Detailed Description

A line, ray, point or closure point.

An object of the class Generator is one of the following:

- a line $\boldsymbol{l} = (a_0, \ldots, a_{n-1})^{\mathrm{T}}$;
- a ray $\boldsymbol{r} = (a_0, \ldots, a_{n-1})^{\mathrm{T}}$;
- a point $\boldsymbol{p} = (\frac{a_0}{d}, \ldots, \frac{a_{n-1}}{d})^{\mathrm{T}}$;
- a closure point $\boldsymbol{c} = (\frac{a_0}{d}, \ldots, \frac{a_{n-1}}{d})^{\mathrm{T}}$;

where n is the dimension of the space and, for points and closure points, $d > 0$ is the divisor.

A note on terminology.

As observed in Section Representations of Convex Polyhedra, there are cases when, in order to represent a polyhedron $\mathcal{P}$ using the generator system $\mathcal{G} = (L, R, P, C)$, we need to include in the finite set P even points of $\mathcal{P}$ that are *not* vertices of $\mathcal{P}$. This situation is even more frequent when working with NNC polyhedra and it is the reason why we prefer to use the word 'point' where other libraries use the word 'vertex'.

How to build a generator.

Each type of generator is built by applying the corresponding function (`line`, `ray`, `point` or `closure_point`) to a linear expression, representing a direction in the space; the space-dimension of the generator is defined as the space-dimension of the corresponding linear expression. Linear expressions used to define a generator should be homogeneous (any constant term will be simply ignored). When defining points and closure points, an optional Integer argument can be used as a common *divisor* for all the coefficients occurring in the provided linear expression; the default value for this argument is 1. In all the following examples it is assumed that variables `x`, `y` and `z` are defined as follows:

```
Variable x(0);
Variable y(1);
Variable z(2);
```

Example 1

The following code builds a line with direction $x - y - z$ and having space-dimension 3:

```
Generator l = line(x - y - z);
```

As mentioned above, the constant term of the linear expression is not relevant. Thus, the following code has the same effect:

```
Generator l = line(x - y - z + 15);
```

By definition, the origin of the space is not a line, so that the following code throws an exception:

```
Generator l = line(0*x);
```

Example 2

The following code builds a ray with the same direction as the line in Example 1:

```
Generator r = ray(x - y - z);
```

As is the case for lines, when specifying a ray the constant term of the linear expression is not relevant; also, an exception is thrown when trying to build a ray from the origin of the space.

Example 3

The following code builds the point $\boldsymbol{p} = (1, 0, 2)^{\mathrm{T}} \in \mathbb{R}^3$:

```
Generator p = point(1*x + 0*y + 2*z);
```

The same effect can be obtained by using the following code:

```
Generator p = point(x + 2*z);
```

Similarly, the origin $\mathbf{0} \in \mathbb{R}^3$ can be defined using either one of the following lines of code:

```
Generator origin3 = point(0*x + 0*y + 0*z);
Generator origin3_alt = point(0*z);
```

Note however that the following code would have defined a different point, namely $\mathbf{0} \in \mathbb{R}^2$:

```
Generator origin2 = point(0*y);
```

The following two lines of code both define the only point having space-dimension zero, namely $\mathbf{0} \in \mathbb{R}^0$. In the second case we exploit the fact that the first argument of the function `point` is optional.

```
Generator origin0 = Generator::zero_dim_point();
Generator origin0_alt = point();
```

Example 4

The point $\boldsymbol{p}$ specified in Example 3 above can also be obtained with the following code, where we provide a non-default value for the second argument of the function `point` (the divisor):

```
Generator p = point(2*x + 0*y + 4*z, 2);
```

Obviously, the divisor can be usefully exploited to specify points having some non-integer (but rational) coordinates. For instance, the point $\boldsymbol{q} = (-1.5, 3.2, 2.1)^{\mathrm{T}} \in \mathbb{R}^3$ can be specified by the following code:

```
Generator q = point(-15*x + 32*y + 21*z, 10);
```

If a zero divisor is provided, an exception is thrown.

Example 5

Closures points are specified in the same way we defined points, but invoking their specific constructor function. For instance, the closure point $\boldsymbol{c} = (1, 0, 2)^{\mathrm{T}} \in \mathbb{R}^3$ is defined by

```
Generator c = closure_point(1*x + 0*y + 2*z);
```

For the particular case of the (only) closure point having space-dimension zero, we can use any of the following:

```
Generator closure_origin0 = Generator::zero_dim_closure_point();
Generator closure_origin0_alt = closure_point();
```

How to inspect a generator

Several methods are provided to examine a generator and extract all the encoded information: its space-dimension, its type and the value of its integer coefficients.

Example 6

The following code shows how it is possible to access each single coefficient of a generator. If `g1` is a point having coordinates $(a_0, \ldots, a_{n-1})^{\mathrm{T}}$, we construct the closure point `g2` having coordinates $(a_0, 2a_1, \ldots, (i+1)a_i, \ldots, na_{n-1})^{\mathrm{T}}$.

```
if (g1.is_point()) {
  cout << "Point g1: " << g1 << endl;
  LinExpression e;
  for (int i = g1.space_dimension() - 1; i >= 0; i--)
    e += (i + 1) * g1.coefficient(Variable(i)) * Variable(i);
  Generator g2 = closure_point(e, g1.divisor());
  cout << "Closure point g2: " << g2 << endl;
}
else
  cout << "Generator g1 is not a point." << endl;
```

Therefore, for the point

```
Generator g1 = point(2*x - y + 3*z, 2);
```

we would obtain the following output:

```
Point g1: p((2*A - B + 3*C)/2)
Closure point g2: cp((2*A - 2*B + 9*C)/2)
```

When working with (closure) points, be careful not to confuse the notion of *coefficient* with the notion of *coordinate*: these are equivalent only when the divisor of the (closure) point is 1.

8.4.2 Member Enumeration Documentation

8.4.2.1 enum Parma_Polyhedra_Library::Generator::Type

The generator type.

Enumeration values:

LINE The generator is a line.

RAY The generator is a ray.

POINT The generator is a point.

CLOSURE_POINT The generator is a closure point.

8.4.3 Member Function Documentation

8.4.3.1 Generator Parma_Polyhedra_Library::Generator::line (const LinExpression & *e*) `[static]`

Returns the line of direction `e`.

Exceptions:

std::invalid_argument thrown if the homogeneous part of `e` represents the origin of the vector space.

8.4.3.2 Generator Parma_Polyhedra_Library::Generator::ray (const LinExpression & *e*) `[static]`

Returns the ray of direction `e`.

Exceptions:

std::invalid_argument thrown if the homogeneous part of `e` represents the origin of the vector space.

8.4.3.3 Generator Parma_Polyhedra_Library::Generator::point (const LinExpression & *e* = LinExpression::zero(), const Integer & *d* = Integer_one()) `[static]`

Returns the point at `e` / `d`.

Both `e` and `d` are optional arguments, with default values LinExpression::zero() and Integer_one(), respectively.

Exceptions:

std::invalid_argument thrown if `d` is zero.

8.4.3.4 Generator Parma_Polyhedra_Library::Generator::closure_point (const LinExpression & *e* = LinExpression::zero(), const Integer & *d* = Integer_one()) `[static]`

Returns the closure point at `e` / `d`.

Both `e` and `d` are optional arguments, with default values LinExpression::zero() and Integer_one(), respectively.

Exceptions:

std::invalid_argument thrown if `d` is zero.

8.4.3.5 const Integer& Parma_Polyhedra_Library::Generator::coefficient (Variable *v*) const

Returns the coefficient of `v` in ∗`this`.

Exceptions:

std::invalid_argument thrown if the index of `v` is greater than or equal to the space-dimension of ∗`this`.

8.4.3.6 const Integer& Parma_Polyhedra_Library::Generator::divisor () const

If ∗`this` is either a point or a closure point, returns its divisor.

Exceptions:

std::invalid_argument thrown if ∗`this` is neither a point nor a closure point.

8.5 Parma_Polyhedra_Library::LinExpression Class Reference

A linear expression.

Public Methods

- LinExpression ()

  Default constructor: returns a copy of LinExpression::zero().

- LinExpression (const LinExpression &e)

  Ordinary copy-constructor.

- virtual ∼LinExpression ()

  Destructor.

- LinExpression (const Integer &n)

  Builds the linear expression corresponding to the inhomogeneous term `n`.

- LinExpression (const Variable &v)

  Builds the linear expression corresponding to the variable `v`.

- LinExpression (const Constraint &c)

  Builds the linear expression corresponding to constraint `c`.

- LinExpression (const Generator &g)

  Builds the linear expression corresponding to generator `g` *(for points and closure points, the divisor is not copied).*

- size_t space_dimension () const

  Returns the dimension of the vector space enclosing ∗`this`.

Static Public Methods

- const LinExpression & zero ()

  Returns the (zero-dimension space) constant 0.

Friends

- class Parma_Polyhedra_Library::Constraint
- class Parma_Polyhedra_Library::Generator
- class Parma_Polyhedra_Library::Polyhedron
- LinExpression Parma_Polyhedra_Library::operator+ (const LinExpression &e1, const LinExpression &e2)

  Returns the linear expression `e1 + e2`.

- LinExpression Parma_Polyhedra_Library::operator+ (const Integer &n, const LinExpression &e)

  Returns the linear expression `n + e`.

- LinExpression Parma_Polyhedra_Library::operator+ (const LinExpression &e, const Integer &n)

  Returns the linear expression `e + n`.

- LinExpression Parma_Polyhedra_Library::operator- (const LinExpression &e)

  Returns the linear expression `- e`.

- LinExpression Parma_Polyhedra_Library::operator- (const LinExpression &e1, const LinExpression &e2)

  Returns the linear expression `e1 - e2`.

- LinExpression Parma_Polyhedra_Library::operator- (const Integer &n, const LinExpression &e)

  Returns the linear expression `n - e`.

- LinExpression Parma_Polyhedra_Library::operator- (const LinExpression &e, const Integer &n)

  Returns the linear expression `e - n`.

- LinExpression Parma_Polyhedra_Library::operator * (const Integer &n, const LinExpression &e)

  Returns the linear expression `n * e`.

- LinExpression Parma_Polyhedra_Library::operator * (const LinExpression &e, const Integer &n)

  Returns the linear expression `e * n`.

- LinExpression & Parma_Polyhedra_Library::operator+= (LinExpression &e1, const LinExpression &e2)

  Returns the linear expression `e1 + e2` *and assigns it to* `e1`.

- LinExpression & Parma_Polyhedra_Library::operator+= (LinExpression &e, const Variable &v)

  Returns the linear expression `e + v` *and assigns it to* `e`.

- LinExpression & Parma_Polyhedra_Library::operator+= (LinExpression &e, const Integer &n)

  Returns the linear expression `e + n` *and assigns it to* `e`.

- LinExpression & Parma_Polyhedra_Library::operator-= (LinExpression &e1, const LinExpression &e2)

  Returns the linear expression `e1 - e2` *and assigns it to* `e1`.

- LinExpression & Parma_Polyhedra_Library::operator-= (LinExpression &e, const Variable &v)

  Returns the linear expression `e - v` *and assigns it to* `e`.

- LinExpression & Parma_Polyhedra_Library::operator-= (LinExpression &e, const Integer &n)

  Returns the linear expression `e - n` *and assigns it to* `e`.

Related Functions

(Note that these are not member functions.)

- void swap (Parma_Polyhedra_Library::LinExpression &x, Parma_Polyhedra_Library::LinExpression &y)

  Specializes `std::swap`.

8.5.1 Detailed Description

A linear expression.

An object of the class LinExpression represents the linear expression

$$\sum_{i=0}^{n-1} a_i x_i + b$$

where n is the dimension of the space, each a_i is the integer coefficient of the `i` -th variable x_i and b is the integer for the inhomogeneous term.

How to build a linear expression.

Linear expressions are the basic blocks for defining both constraints (i.e., linear equalities or inequalities) and generators (i.e., lines, rays, points and closure points). A full set of functions is defined to provide a convenient interface for building complex linear expressions starting from simpler ones and from objects of the classes Variable and Integer: available operators include unary negation, binary addition and subtraction, as well as multiplication by an Integer. The space-dimension of a linear expression is defined as the maximum space-dimension of the arguments used to build it: in particular, the space-dimension of a Variable `x` is defined as `x.id()+1`, whereas all the objects of the class Integer have space-dimension zero.

Example

The following code builds the linear expression $4x - 2y - z + 14$, having space-dimension 3:

```
LinExpression e = 4*x - 2*y - z + 14;
```

Another way to build the same linear expression is:

```
LinExpression e1 = 4*x;
LinExpression e2 = 2*y;
LinExpression e3 = z;
LinExpression e = LinExpression(14);
e += e1 - e2 - e3;
```

Note that `e1`, `e2` and `e3` have space-dimension 1, 2 and 3, respectively; also, in the fourth line of code, `e` is created with space-dimension zero and then extended to space-dimension 3.

8.5.2 Constructor & Destructor Documentation

8.5.2.1 Parma_Polyhedra_Library::LinExpression::LinExpression (const Constraint & *c*) [explicit]

Builds the linear expression corresponding to constraint `c`.

Given the constraint $c = \left(\sum_{i=0}^{n-1} a_i x_i + b \bowtie 0\right)$, where $\bowtie \in \{=, \geq, >\}$, builds the linear expression $\sum_{i=0}^{n-1} a_i x_i + b$. If `c` is an inequality (resp., equality) constraint, then the built linear expression is unique up to a positive (resp., non-zero) factor.

8.5.2.2 Parma_Polyhedra_Library::LinExpression::LinExpression (const Generator & *g*) [explicit]

Builds the linear expression corresponding to generator `g` (for points and closure points, the divisor is not copied).

Given the generator $g = (\frac{a_0}{d}, \ldots, \frac{a_{n-1}}{d})^{\mathrm{T}}$ (where, for lines and rays, we have $d = 1$), builds the linear expression $\sum_{i=0}^{n-1} a_i x_i$. The inhomogeneous term of the linear expression will always be 0. If `g` is a ray, point or closure point (resp., a line), then the linear expression is unique up to a positive (resp., non-zero) factor.

8.6 Parma_Polyhedra_Library::NNC_Polyhedron Class Reference

A not necessarily closed convex polyhedron.

Inherits Parma_Polyhedra_Library::Polyhedron.

Public Methods

- NNC_Polyhedron (size_t num_dimensions=0, Degenerate_Kind kind=UNIVERSE)

  Builds either the universe or the empty NNC polyhedron.

- NNC_Polyhedron (ConSys &cs)

  Builds a NNC polyhedron from a system of constraints.

- NNC_Polyhedron (GenSys &gs)

  Builds a NNC polyhedron from a system of generators.

- NNC_Polyhedron (const C_Polyhedron &y)

  Builds a NNC polyhedron from the C_Polyhedron `y`.

- template<class Box> NNC_Polyhedron (const Box &box, From_Bounding_Box dummy)

  Builds an NNC polyhedron out of a generic, interval-based bounding box.

- NNC_Polyhedron (const NNC_Polyhedron &y)

  Ordinary copy-constructor.

- NNC_Polyhedron & operator= (const NNC_Polyhedron &y)

  The assignment operator. (`*this` *and* `y` *can be dimension-incompatible.).*

- ~NNC_Polyhedron ()

Destructor.

8.6.1 Detailed Description

A not necessarily closed convex polyhedron.

An object of the class NNC_Polyhedron represents a *not necessarily closed* (NNC) convex polyhedron in the vector space $\mathbb{R}^n$.

Note:

Since NNC polyhedra are a generalization of closed polyhedra, any object of the class C_Polyhedron can be (explicitly) converted into an object of the class NNC_Polyhedron. The reason for defining two different classes is that objects of the class C_Polyhedron are characterized by a more efficient implementation, requiring less time and memory resources.

8.6.2 Constructor & Destructor Documentation

8.6.2.1 Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (size_t *num_dimensions* = 0, Degenerate_Kind *kind* = UNIVERSE) `[explicit]`

Builds either the universe or the empty NNC polyhedron.

Parameters:

num_dimensions The number of dimensions of the vector space enclosing the NNC polyhedron.

kind Specifies whether a universe or an empty NNC polyhedron should be built.

Both parameters are optional: by default, a 0-dimension space universe NNC polyhedron is built.

8.6.2.2 Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (ConSys & *cs*)

Builds a NNC polyhedron from a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

cs The system of constraints defining the polyhedron. It is not declared `const` because it can be modified.

8.6.2.3 Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (GenSys & *gs*)

Builds a NNC polyhedron from a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

gs The system of generators defining the polyhedron. It is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if the system of generators is not empty but has no points.

8.6.2.4 template<class Box> Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (const Box & *box*, From_Bounding_Box *dummy*)

Builds an NNC polyhedron out of a generic, interval-based bounding box.

For a description of the methods that should be provided by the template class Box, see the documentation of the protected method: template <class Box> Polyhedron::Polyhedron(Topology topol, const Box& box);

Parameters:

box The bounding box representing the polyhedron to be built.

dummy A dummy tag to syntactically differentiate this one from the other constructors.

8.7 Parma_Polyhedra_Library::Poly_Con_Relation Class Reference

The relation between a polyhedron and a constraint.

Public Methods

- bool implies (const Poly_Con_Relation &y) const

  True if and only if `*this` *implies* `y`*.*

- bool OK () const

  Checks if all the invariants are satisfied.

Static Public Methods

- Poly_Con_Relation nothing ()

  The assertion that says nothing.

- Poly_Con_Relation is_disjoint ()

  The polyhedron and the set of points satisfying the constraint are disjoint.

- Poly_Con_Relation strictly_intersects ()

  The polyhedron intersects the set of points satisfying the constraint, but it is not included in it.

- Poly_Con_Relation is_included ()

  The polyhedron is included in the set of points satisfying the constraint.

- Poly_Con_Relation saturates ()

  The polyhedron is included in the set of points saturating the constraint.

Friends

- bool Parma_Polyhedra_Library::operator== (const Poly_Con_Relation &x, const Poly_Con_Relation &y)

  True if and only if `x` *and* `y` *are logically equivalent.*

- bool Parma_Polyhedra_Library::operator!= (const Poly_Con_Relation &x, const Poly_Con_Relation &y)

  True if and only if x *and* y *are not logically equivalent.*

- Poly_Con_Relation Parma_Polyhedra_Library::operator && (const Poly_Con_Relation &x, const Poly_Con_Relation &y)

  Yields the logical conjunction of x *and* y.

- Poly_Con_Relation Parma_Polyhedra_Library::operator- (const Poly_Con_Relation &x, const Poly_-Con_Relation &y)

  Yields the assertion with all the conjuncts of x *that are not in* y.

- std::ostream & Parma_Polyhedra_Library::operator<< (std::ostream &s, const Poly_Con_Relation &r)

  Output operator.

8.7.1 Detailed Description

The relation between a polyhedron and a constraint.

This class implements conjunctions of assertions on the relation between a polyhedron and a constraint.

8.8 Parma_Polyhedra_Library::Poly_Gen_Relation Class Reference

The relation between a polyhedron and a generator.

Public Methods

- bool implies (const Poly_Gen_Relation &y) const

  True if and only if ∗this *implies* y.

- bool OK () const

  Checks if all the invariants are satisfied.

Static Public Methods

- Poly_Gen_Relation nothing ()

  The assertion that says nothing.

- Poly_Gen_Relation subsumes ()

  Adding the generator would not change the polyhedron.

Friends

- bool Parma_Polyhedra_Library::operator== (const Poly_Gen_Relation &x, const Poly_Gen_Relation &y)

True if and only if x *and* y *are logically equivalent.*

- bool Parma_Polyhedra_Library::operator!= (const Poly_Gen_Relation &x, const Poly_Gen_Relation &y)

  True if and only if x *and* y *are not logically equivalent.*

- Poly_Gen_Relation Parma_Polyhedra_Library::operator && (const Poly_Gen_Relation &x, const Poly_Gen_Relation &y)

  Yields the logical conjunction of x *and* y.

- Poly_Gen_Relation Parma_Polyhedra_Library::operator- (const Poly_Gen_Relation &x, const Poly_Gen_Relation &y)

  Yields the assertion with all the conjuncts of x *that are not in* y.

- std::ostream & Parma_Polyhedra_Library::operator<< (std::ostream &s, const Poly_Gen_Relation &r)

  Output operator.

8.8.1 Detailed Description

The relation between a polyhedron and a generator.

This class implements conjunctions of assertions on the relation between a polyhedron and a generator.

8.9 Parma_Polyhedra_Library::Polyhedron Class Reference

The base class for convex polyhedra.

Inherited by Parma_Polyhedra_Library::C_Polyhedron, and Parma_Polyhedra_Library::NNC_Polyhedron.

Public Types

- enum Degenerate_Kind { UNIVERSE, EMPTY }

  Kinds of degenerate polyhedra.

Public Methods

- ∼Polyhedron ()

  Destructor.

- size_t space_dimension () const

  Returns the dimension of the vector space enclosing ∗this.

- bool intersection_assign_and_minimize (const Polyhedron &y)

  Assigns to ∗this *the intersection of* ∗this *and* y*, minimizing the result.*

- void intersection_assign (const Polyhedron &y)

  Assigns to ∗this *the intersection of* ∗this *and* y*. The result is not guaranteed to be minimized.*

- bool poly_hull_assign_and_minimize (const Polyhedron &y)

  Assigns to `*this` *the poly-hull of* `*this` *and* `y`*, minimizing the result.*

- void poly_hull_assign (const Polyhedron &y)

  Assigns to `*this` *the poly-hull* `*this` *and* `y`*. The result is not guaranteed to be minimized.*

- bool poly_difference_assign_and_minimize (const Polyhedron &y)

  Assigns to `*this` *the poly-difference of* `*this` *and* `y`*, minimizing the result.*

- void poly_difference_assign (const Polyhedron &y)

  Assigns to `*this` *the poly-difference of* `*this` *and* `y`*. The result is not guaranteed to be minimized.*

- Poly_Con_Relation relation_with (const Constraint &c) const

  Returns the relations holding between the polyhedron `*this` *and the constraint* `c`*.*

- Poly_Gen_Relation relation_with (const Generator &g) const

  Returns the relations holding between the polyhedron `*this` *and the generator* `g`*.*

- void H79_widening_assign (const Polyhedron &y)

  Assigns to `*this` *the result of computing the H79-widening between* `*this` *and* `y`*.*

- void limited_H79_widening_assign (const Polyhedron &y, ConSys &cs)

  Limits the H79-widening computation between `*this` *and* `y` *by enforcing constraints* `cs` *and assigns the result to* `*this`*.*

- void time_elapse_assign (const Polyhedron &y)

  Assigns to `*this` *the result of computing the time-elapse between* `*this` *and* `y`*.*

- const ConSys & constraints () const

  Returns the system of constraints.

- const ConSys & minimized_constraints () const

  Returns the system of constraints, with no redundant constraint.

- const GenSys & generators () const

  Returns the system of generators.

- const GenSys & minimized_generators () const

  Returns the system of generators, with no redundant generator.

- void add_constraint (const Constraint &c)

  Adds a copy of constraint `c` *to the system of constraints of* `*this`*.*

- void add_generator (const Generator &g)

  Adds a copy of generator `g` *to the system of generators of* `*this`*.*

- void affine_image (const Variable &var, const LinExpression &expr, const Integer &denominator=Integer_one())

Assigns to `*this` *the affine image of* `*this` *under the function mapping variable* `v` *into the affine expression specified by* `expr` *and* `d`.

- void affine_preimage (const Variable &var, const LinExpression &expr, const Integer &denominator=Integer_one())

  Assigns to `*this` *the affine preimage of* `*this` *under the function mapping variable* `v` *into the affine expression specified by* `expr` *and* `d`.

- template<class Box> void shrink_bounding_box (Box &box) const

  Use `*this` *to shrink a generic, interval-based bounding box.*

- bool OK (bool check_not_empty=false) const

  Checks if all the invariants are satisfied.

- void add_dimensions_and_embed (size_t dim)

  Adds `dim` *new dimensions and embeds the old polyhedron into the new space.*

- void add_dimensions_and_project (size_t dim)

  Adds `dim` *new dimensions to the polyhedron and does not embed it in the new space.*

- void remove_dimensions (const std::set< Variable > &to_be_removed)

  Removes all the specified dimensions.

- void remove_higher_dimensions (size_t new_dimension)

  Removes the higher dimensions so that the resulting space will have dimension `new_dimension`.

- bool add_constraints_and_minimize (ConSys &cs)

  Adds the specified constraints and minimizes the result, which is assigned to `*this`.

- void add_constraints (ConSys &cs)

  Adds the specified constraints without minimizing.

- void add_dimensions_and_constraints (ConSys &cs)

  First increases the space dimension of `*this` *by adding* `cs.space_dimension()` *new dimensions; then adds to the system of constraints of* `*this` *a renamed-apart version of the constraints in* `cs`.

- bool add_generators_and_minimize (GenSys &gs)

  Adds the specified generators and minimizes the result, which is assigned to `*this`.

- void add_generators (GenSys &gs)

  Adds the specified generators without minimizing.

- bool check_empty () const

  Returns `true` *if and only if* `*this` *is an empty polyhedron.*

- bool check_universe () const

  Returns `true` *if and only if* `*this` *is a universe polyhedron.*

- bool is_bounded () const

  Returns `true` *if and only if* `*this` *is a bounded polyhedron.*

- bool bounds_from_above (const LinExpression &expr) const

  Returns `true` *if and only if* `expr` *is bounded from above in* `*this`.

- bool bounds_from_below (const LinExpression &expr) const

  Returns `true` *if and only if* `expr` *is bounded from below in* `*this`.

- bool is_topologically_closed () const

  Returns `true` *if and only if* `*this` *is a topologically closed subset of the vector space.*

- void topological_closure_assign ()

  Assigns to `*this` *its topological closure.*

- void swap (Polyhedron &y)

  Swaps `*this` *with polyhedron* `y`. *(*`*this` *and* `y` *can be dimension-incompatible.).*

Protected Methods

- Polyhedron (const Polyhedron &y)

  Ordinary copy-constructor.

- Polyhedron (Topology topol, size_t num_dimensions, Degenerate_Kind kind)

  Builds a polyhedron having the specified properties.

- Polyhedron (Topology topol, ConSys &cs)

  Builds a polyhedron from a system of constraints.

- Polyhedron (Topology topol, GenSys &gs)

  Builds a polyhedron from a system of generators.

- template<class Box> Polyhedron (Topology topol, const Box &box)

  Builds a polyhedron out of a generic, interval-based bounding box.

- Polyhedron & operator= (const Polyhedron &y)

  The assignment operator. (`*this` *and* `y` *can be dimension-incompatible.).*

Friends

- bool Parma_Polyhedra_Library::operator<= (const Polyhedron &x, const Polyhedron &y)

  Returns `true` *if and only if polyhedron* `x` *is contained in polyhedron* `y`.

- std::ostream & Parma_Polyhedra_Library::operator<< (std::ostream &s, const Polyhedron &p)

  Output operator.

- std::istream & Parma_Polyhedra_Library::operator>> (std::istream &s, Polyhedron &p)

  Input operator.

Related Functions

(Note that these are not member functions.)

- bool operator== (const Polyhedron &x, const Polyhedron &y)
  Returns `true` *if and only if* `x` *and* `y` *are the same polyhedron.*
- bool operator!= (const Polyhedron &x, const Polyhedron &y)
  Returns `true` *if and only if* `x` *and* `y` *are different polyhedra.*
- bool operator< (const Polyhedron &x, const Polyhedron &y)
  Returns `true` *if and only if* `x` *is strictly contained in* `y`.
- bool operator> (const Polyhedron &x, const Polyhedron &y)
  Returns `true` *if and only if* `x` *strictly contains* `y`.
- bool operator>= (const Polyhedron &x, const Polyhedron &y)
  Returns `true` *if and only if* `x` *contains* `y`.
- void swap (Parma_Polyhedra_Library::Polyhedron &x, Parma_Polyhedra_Library::Polyhedron &y)
  Specializes `std::swap`.

8.9.1 Detailed Description

The base class for convex polyhedra.

An object of the class Polyhedron represents a convex polyhedron in the vector space $\mathbb{R}^n$.

A polyhedron can be specified as either a finite system of constraints or a finite system of generators (see Section Representations of Convex Polyhedra) and it is always possible to obtain either representation. That is, if we know the system of constraints, we can obtain from this the system of generators that define the same polyhedron and vice versa. These systems can contain redundant members: in this case we say that they are not in the minimal form. Most operators on polyhedra are provided with two implementations: one of these, denoted <`operator-name`>`_and_minimize`, also enforces the minimization of the representations, and returns the Boolean value `false` whenever the resulting polyhedron turns out to be empty.

Two key attributes of any polyhedron are its topological kind (recording whether it is a C_Polyhedron or an NNC_Polyhedron object) and its space dimension (the dimension $n \in \mathbb{N}$ of the enclosing vector space):

- all polyhedra, the empty ones included, are endowed with a specific topology and space dimension;
- most operations working on a polyhedron and another object (i.e., another polyhedron, a constraint or generator, a set of variables, etc.) will throw an exception if the polyhedron and the object are not both topology-compatible and dimension-compatible (see Section Representations of Convex Polyhedra);
- there is no way to change the topology of a polyhedron; rather, there are constructors of the two derived classes that builds a new polyhedron having a topology when provided with the corresponding polyhedron of the other topology;
- the only ways to change the space dimension of a polyhedron are:
  - *explicit* calls to operators provided for that purpose;
  - standard copy, assignment and swap operators.

Note that four different polyhedra can be defined on the zero-dimension space: the empty polyhedron, either closed or NNC, and the universe polyhedron R^0, again either closed or NNC.

In all the examples it is assumed that variables x and y are defined (where they are used) as follows:

```
Variable x(0);
Variable y(1);
```

Example 1

The following code builds a polyhedron corresponding to a square in $\mathbb{R}^2$, given as a system of constraints:

```
ConSys cs;
cs.add_constraint(x >= 0);
cs.add_constraint(x <= 3);
cs.add_constraint(y >= 0);
cs.add_constraint(y <= 3);
Polyhedron ph(cs);
```

The following code builds the same polyhedron as above, but starting from a system of generators specifying the four vertices of the square:

```
GenSys gs;
gs.add_generator(point(0*x + 0*y));
gs.add_generator(point(0*x + 3*y));
gs.add_generator(point(3*x + 0*y));
gs.add_generator(point(3*x + 3*y));
Polyhedron ph(gs);
```

Example 2

The following code builds an unbounded polyhedron corresponding to a half-strip in $\mathbb{R}^2$, given as a system of constraints:

```
ConSys cs;
cs.add_constraint(x >= 0);
cs.add_constraint(x - y <= 0);
cs.add_constraint(x - y + 1 >= 0);
Polyhedron ph(cs);
```

The following code builds the same polyhedron as above, but starting from the system of generators specifying the two vertices of the polyhedron and one ray:

```
GenSys gs;
gs.add_generator(point(0*x + 0*y));
gs.add_generator(point(0*x + y));
gs.add_generator(ray(x - y));
Polyhedron ph(gs);
```

Example 3

The following code builds the polyhedron corresponding to an half-plane by adding a single constraint to the universe polyhedron in $\mathbb{R}^2$:

```
Polyhedron ph(2);
ph.add_constraint(y >= 0);
```

The following code builds the same polyhedron as above, but starting from the empty polyhedron in the space $\mathbb{R}^2$ and inserting the appropriate generators (a point, a ray and a line).

```
Polyhedron ph(2, Polyhedron::EMPTY);
ph.add_generator(point(0*x + 0*y));
ph.add_generator(ray(y));
ph.add_generator(line(x));
```

Note that, although the above polyhedron has no vertices, we must add one point, because otherwise the result of the Minkowsky's sum would be an empty polyhedron. To avoid subtle errors related to the minimization process, it is required that the first generator inserted in an empty polyhedron is a point (otherwise, an exception is thrown).

Example 4

The following code shows the use of the function `add_dimensions_and_embed`:

```
Polyhedron ph(1);
ph.add_constraint(x == 2);
ph.add_dimensions_and_embed(1);
```

We build the universe polyhedron in the 1-dimension space $\mathbb{R}$. Then we add a single equality constraint, thus obtaining the polyhedron corresponding to the singleton set $\{2\} \subseteq \mathbb{R}$. After the last line of code, the resulting polyhedron is

$$\left\{ (2, y)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; y \in \mathbb{R} \right\}.$$

Example 5

The following code shows the use of the function `add_dimensions_and_project`:

```
Polyhedron ph(1);
ph.add_constraint(x == 2);
ph.add_dimensions_and_project(1);
```

The first two lines of code are the same as in Example 4 for `add_dimensions_and_embed`. After the last line of code, the resulting polyhedron is the singleton set $\left\{(2, 0)^{\mathrm{T}}\right\} \subseteq \mathbb{R}^2$.

Example 6

The following code shows the use of the function `affine_image`:

```
Polyhedron ph(2, Polyhedron::EMPTY);
ph.add_generator(point(0*x + 0*y));
ph.add_generator(point(0*x + 3*y));
ph.add_generator(point(3*x + 0*y));
ph.add_generator(point(3*x + 3*y));
LinExpression coeff = x + 4;
ph.affine_image(x, coeff);
```

In this example the starting polyhedron is a square in $\mathbb{R}^2$, the considered variable is x and the affine expression is $x + 4$. The resulting polyhedron is the same square translated towards right. Moreover, if the affine transformation for the same variable `x` is $x + y$:

```
LinExpression coeff = x + y;
```

the resulting polyhedron is a parallelogram with the height equal to the side of the square and the oblique sides parallel to the line $x - y$. Instead, if we do not use an invertible transformation for the same variable; for example, the affine expression y:

```
LinExpression coeff = y;
```

the resulting polyhedron is a diagonal of the square.

Example 7

The following code shows the use of the function `affine_preimage`:

```
Polyhedron ph(2);
ph.add_constraint(x >= 0);
ph.add_constraint(x <= 3);
ph.add_constraint(y >= 0);
ph.add_constraint(y <= 3);
LinExpression coeff = x + 4;
ph.affine_preimage(x, coeff);
```

In this example the starting polyhedron, `var` and the affine expression and the denominator are the same as in Example 6, while the resulting polyhedron is again the same square, but translated towards left. Moreover, if the affine transformation for `x` is $x + y$

```
LinExpression coeff = x + y;
```

the resulting polyhedron is a parallelogram with the height equal to the side of the square and the oblique sides parallel to the line $x + y$. Instead, if we do not use an invertible transformation for the same variable `x`, for example, the affine expression y:

```
LinExpression coeff = y;
```

the resulting polyhedron is a line that corresponds to the y axis.

Example 8

For this example we use also the variables:

```
Variable z(2);
Variable w(3);
```

The following code shows the use of the function `remove_dimensions`:

```
GenSys gs;
gs.add_generator(point(3*x + y +0*z + 2*w));
Polyhedron ph(gs);
set<Variable> to_be_removed;
to_be_removed.insert(y);
to_be_removed.insert(z);
ph.remove_dimensions(to_be_removed);
```

The starting polyhedron is the singleton set $\{(3, 1, 0, 2)^{\mathrm{T}}\} \subseteq \mathbb{R}^4$, while the resulting polyhedron is $\{(3, 2)^{\mathrm{T}}\} \subseteq \mathbb{R}^2$. Be careful when removing dimensions *incrementally*: since dimensions are automatically renamed after each application of the `remove_dimensions` operator, unexpected results can be obtained. For instance, by using the following code we would obtain a different result:

```
set<Variable> to_be_removed1;
to_be_removed1.insert(y);
ph.remove_dimensions(to_be_removed1);
set<Variable> to_be_removed2;
to_be_removed2.insert(z);
ph.remove_dimensions(to_be_removed2);
```

In this case, the result is the polyhedron $\{(3, 0)^{\mathrm{T}}\} \subseteq \mathbb{R}^2$: when removing the set of dimensions `to_be_removed2` we are actually removing variable w of the original polyhedron. For the same reason, the operator `remove_dimensions` is not idempotent: removing twice the same set of dimensions is never a no-op.

8.9.2 Member Enumeration Documentation

8.9.2.1 enum Parma_Polyhedra_Library::Polyhedron::Degenerate_Kind

Kinds of degenerate polyhedra.

Enumeration values:

UNIVERSE The universe polyhedron, i.e., the whole vector space.

EMPTY The empty polyhedron, i.e., the empty set.

8.9.3 Constructor & Destructor Documentation

8.9.3.1 Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, size_t *num_dimensions*, Degenerate_Kind *kind*) `[protected]`

Builds a polyhedron having the specified properties.

Parameters:

topol The topology of the polyhedron;

num_dimensions The number of dimensions of the vector space enclosing the polyhedron;

kind Specifies whether the universe or the empty polyhedron has to be built.

8.9.3.2 Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, ConSys & *cs*) `[protected]`

Builds a polyhedron from a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

topol The topology of the polyhedron;

cs The system of constraints defining the polyhedron. It is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if the topology of `cs` is incompatible with `topology`.

8.9.3.3 Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, GenSys & *gs*) `[protected]`

Builds a polyhedron from a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

topol The topology of the polyhedron;

gs The system of generators defining the polyhedron. It is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if if the topology of `gs` is incompatible with `topol`, or if the system of generators is not empty but has no points.

8.9.3.4 template<class Box> Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, const Box & *box*) `[protected]`

Builds a polyhedron out of a generic, interval-based bounding box.

Parameters:

topol The topology of the polyhedron;

box The bounding box representing the polyhedron to be built.

Exceptions:

std::invalid_argument thrown if `box` has intervals that are incompatible with `topol`.

The template class Box must provide the following methods.

```
unsigned int space_dimension() const
```

returns the dimension of the vector space enclosing the polyhedron represented by the bounding box.

```
bool is_empty() const
```

returns `true` if and only if the bounding box describes the empty set. The `is_empty()` method will always be called before the methods below. However, if `is_empty()` returns `true`, none of the functions below will be called.

```
bool get_lower_bound(unsigned int k, bool closed,
                     Integer& n, Integer& d) const
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from below, simply return `false`. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to `true` if the the lower boundary of I is closed and is set to `false` otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the greatest lower bound of I. The fraction n/d is in canonical form if and only if n and d have no common factors and d is positive, $0/1$ being the unique representation for zero.

```
bool get_upper_bound(unsigned int k, bool closed,
                     Integer& n, Integer& d) const
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from above, simply return `false`. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to `true` if the the upper boundary of I is closed and is set to `false` otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the least upper bound of I.

8.9.4 Member Function Documentation

8.9.4.1 bool Parma_Polyhedra_Library::Polyhedron::intersection_assign_and_minimize (const Polyhedron & *y*)

Assigns to $*$`this` the intersection of $*$`this` and `y`, minimizing the result.

Returns:

`false` if and only if the result is empty.

Exceptions:

std::invalid_argument thrown if $*$`this` and `y` are topology-incompatible or dimension-incompatible.

8.9.4.2 void Parma_Polyhedra_Library::Polyhedron::intersection_assign (const Polyhedron & *y*)

Assigns to $*$`this` the intersection of $*$`this` and `y`. The result is not guaranteed to be minimized.

Exceptions:

std::invalid_argument thrown if $*$`this` and `y` are topology-incompatible or dimension-incompatible.

8.9.4.3 bool Parma_Polyhedra_Library::Polyhedron::poly_hull_assign_and_minimize (const Polyhedron & *y*)

Assigns to ∗this the poly-hull of ∗this and y, minimizing the result.

Returns:
false if and only if the result is empty.

Exceptions:
std::invalid_argument thrown if ∗this and y are topology-incompatible or dimension-incompatible.

8.9.4.4 void Parma_Polyhedra_Library::Polyhedron::poly_hull_assign (const Polyhedron & *y*)

Assigns to ∗this the poly-hull ∗this and y. The result is not guaranteed to be minimized.

Exceptions:
std::invalid_argument thrown if ∗this and y are topology-incompatible or dimension-incompatible.

8.9.4.5 bool Parma_Polyhedra_Library::Polyhedron::poly_difference_assign_and_minimize (const Polyhedron & *y*)

Assigns to ∗this the poly-difference of ∗this and y, minimizing the result.

Returns:
false if and only if the result is empty.

Exceptions:
std::invalid_argument thrown if ∗this and y are topology-incompatible or dimension-incompatible.

8.9.4.6 void Parma_Polyhedra_Library::Polyhedron::poly_difference_assign (const Polyhedron & *y*)

Assigns to ∗this the poly-difference of ∗this and y. The result is not guaranteed to be minimized.

Exceptions:
std::invalid_argument thrown if ∗this and y are topology-incompatible or dimension-incompatible.

8.9.4.7 Poly_Con_Relation Parma_Polyhedra_Library::Polyhedron::relation_with (const Constraint & *c*) const

Returns the relations holding between the polyhedron ∗this and the constraint c.

Exceptions:
std::invalid_argument thrown if ∗this and constraint c are dimension-incompatible.

8.9.4.8 Poly_Gen_Relation Parma_Polyhedra_Library::Polyhedron::relation_with (const Generator & *g*) const

Returns the relations holding between the polyhedron ∗this and the generator g.

Exceptions:
std::invalid_argument thrown if ∗this and generator g are dimension-incompatible.

8.9.4.9 void Parma_Polyhedra_Library::Polyhedron::H79_widening_assign (const Polyhedron & *y*)

Assigns to `*this` the result of computing the H79-widening between `*this` and `y`.

Parameters:

y A polyhedron that *must* be contained in `*this`.

Exceptions:

std::invalid_argument thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

8.9.4.10 void Parma_Polyhedra_Library::Polyhedron::limited_H79_widening_assign (const Polyhedron & *y*, ConSys & *cs*)

Limits the H79-widening computation between `*this` and `y` by enforcing constraints `cs` and assigns the result to `*this`.

Parameters:

y A polyhedron that *must* be contained in `*this`.

cs The system of constraints that limits the widened polyhedron. It is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if `*this`, `y` and `cs` are topology-incompatible or dimension-incompatible.

8.9.4.11 void Parma_Polyhedra_Library::Polyhedron::time_elapse_assign (const Polyhedron & *y*)

Assigns to `*this` the result of computing the time-elapse between `*this` and `y`.

Exceptions:

std::invalid_argument thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

8.9.4.12 void Parma_Polyhedra_Library::Polyhedron::add_constraint (const Constraint & *c*)

Adds a copy of constraint `c` to the system of constraints of `*this`.

Exceptions:

std::invalid_argument thrown if `*this` and constraint `c` are topology-incompatible or dimension-incompatible.

8.9.4.13 void Parma_Polyhedra_Library::Polyhedron::add_generator (const Generator & *g*)

Adds a copy of generator `g` to the system of generators of `*this`.

Exceptions:

std::invalid_argument thrown if `*this` and generator `g` are topology-incompatible or dimension-incompatible, or if `*this` is an empty polyhedron and `g` is not a point.

8.9.4.14 void Parma_Polyhedra_Library::Polyhedron::affine_image (const Variable & *var*, const LinExpression & *expr*, const Integer & *denominator* = Integer_one())

Assigns to ∗`this` the affine image of ∗`this` under the function mapping variable `v` into the affine expression specified by `expr` and `d`.

Parameters:

var The variable to which the affine expression is assigned.

expr The numerator of the affine expression.

denominator The denominator of the affine expression (optional argument with default value 1.)

Exceptions:

std::invalid_argument thrown if `denominator` is zero or if `expr` and ∗`this` are dimension-incompatible or if `var` is not a dimension of ∗`this`.

8.9.4.15 void Parma_Polyhedra_Library::Polyhedron::affine_preimage (const Variable & *var*, const LinExpression & *expr*, const Integer & *denominator* = Integer_one())

Assigns to ∗`this` the affine preimage of ∗`this` under the function mapping variable `v` into the affine expression specified by `expr` and `d`.

Parameters:

var The variable to which the affine expression is substituted.

expr The numerator of the affine expression.

denominator The denominator of the affine expression (optional argument with default value 1.)

Exceptions:

std::invalid_argument thrown if `denominator` is zero or if `expr` and ∗`this` are dimension-incompatible or if `var` is not a dimension of ∗`this`.

8.9.4.16 template<class Box> void Parma_Polyhedra_Library::Polyhedron::shrink_bounding_box (Box & *box*) const

Use ∗`this` to shrink a generic, interval-based bounding box.

Parameters:

box The bounding box to be shrunk.

The template class Box must provide the following methods, whose return value, if any, is simply ignored.

```
set_empty()
```

causes the box to become empty, i.e., to represent the empty set.

```
raise_lower_bound(unsigned int k, bool closed,
                  const Integer& n, const Integer& d)
```

intersects the interval corresponding to the `k`-th dimension with $[n/d, +\infty)$ if `closed` is `true`, with $(n/d, +\infty)$ if `closed` is `false`. The fraction n/d is in canonical form, that is, n and d have no common factors and d is positive, $0/1$ being the unique representation for zero.

```
lower_upper_bound(unsigned int k, bool closed,
                  const Integer& n, const Integer& d)
```

intersects the interval corresponding to the k-th dimension with $(-\infty, n/d]$ if `closed` is `true`, with $(-\infty, n/d)$ if `closed` is `false`. The fraction n/d is in canonical form.

8.9.4.17 bool Parma_Polyhedra_Library::Polyhedron::OK (bool *check_not_empty* = false) const

Checks if all the invariants are satisfied.

Parameters:

check_not_empty `true` if and only if, in addition to checking the invariants, $*$`this` must be checked to be not empty.

Returns:

`true` if and only if $*$`this` satisfies all the invariants and either `check_not_empty` is `false` or $*$`this` is not empty.

The check is performed so as to intrude as little as possible. In case invariants are violated error messages are written on `std::cerr`. This is useful for the purpose of debugging the library.

8.9.4.18 void Parma_Polyhedra_Library::Polyhedron::add_dimensions_and_embed (size_t *dim*)

Adds `dim` new dimensions and embeds the old polyhedron into the new space.

Parameters:

dim The number of dimensions to add.

The new dimensions will be those having the highest indexes in the new polyhedron, which is characterized by a system of constraints in which the variables running through the new dimensions are not constrained. For instance, when starting from the polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and adding a third dimension, the result will be the polyhedron

$$\left\{ (x, y, z)^{\mathrm{T}} \in \mathbb{R}^3 \;\middle|\; (x, y)^{\mathrm{T}} \in \mathcal{P} \right\}.$$

8.9.4.19 void Parma_Polyhedra_Library::Polyhedron::add_dimensions_and_project (size_t *dim*)

Adds `dim` new dimensions to the polyhedron and does not embed it in the new space.

Parameters:

dim The number of dimensions to add.

The new dimensions will be those having the highest indexes in the new polyhedron, which is characterized by a system of constraints in which the variables running through the new dimensions are all constrained to be equal to 0. For instance, when starting from the polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and adding a third dimension, the result will be the polyhedron

$$\left\{ (x, y, 0)^{\mathrm{T}} \in \mathbb{R}^3 \;\middle|\; (x, y)^{\mathrm{T}} \in \mathcal{P} \right\}.$$

8.9.4.20 void Parma_Polyhedra_Library::Polyhedron::remove_dimensions (const std::set< Variable > & *to_be_removed*)

Removes all the specified dimensions.

Parameters:

to_be_removed The set of Variable objects corresponding to the dimensions to be removed.

Exceptions:

std::invalid_argument thrown if `*this` is dimension-incompatible with one of the Variable objects contained in `to_be_removed`.

8.9.4.21 void Parma_Polyhedra_Library::Polyhedron::remove_higher_dimensions (size_t *new_dimension*)

Removes the higher dimensions so that the resulting space will have dimension `new_dimension`.

Exceptions:

std::invalid_argument thrown if `new_dimensions` is greater than the space dimension of `*this`.

8.9.4.22 bool Parma_Polyhedra_Library::Polyhedron::add_constraints_and_minimize (ConSys & *cs*)

Adds the specified constraints and minimizes the result, which is assigned to `*this`.

Returns:

`false` if and only if the result is empty.

Parameters:

cs The constraints that will be added to the current system of constraints. This parameter is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if `*this` and `cs` are topology-incompatible or dimension-incompatible.

8.9.4.23 void Parma_Polyhedra_Library::Polyhedron::add_constraints (ConSys & *cs*)

Adds the specified constraints without minimizing.

Parameters:

cs The constraints that will be added to the current system of constraints. This parameter is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if `*this` and `cs` are topology-incompatible or dimension-incompatible.

8.9.4.24 void Parma_Polyhedra_Library::Polyhedron::add_dimensions_and_constraints (ConSys & *cs*)

First increases the space dimension of ∗`this` by adding `cs.space_dimension()` new dimensions; then adds to the system of constraints of ∗`this` a renamed-apart version of the constraints in `cs`.

Exceptions:

std::invalid_argument thrown if ∗`this` and `cs` are topology-incompatible.

8.9.4.25 bool Parma_Polyhedra_Library::Polyhedron::add_generators_and_minimize (GenSys & *gs*)

Adds the specified generators and minimizes the result, which is assigned to ∗`this`.

Returns:

`false` if and only if the result is empty.

Parameters:

gs The generators that will be added to the current system of generators. The parameter is not declared `const` because it can be modified.

Returns:

`false` if the resulting polyhedron is empty.

Exceptions:

std::invalid_argument thrown if ∗`this` and `gs` are topology-incompatible or dimension-incompatible, or if ∗`this` is empty and the the system of generators `gs` is not empty, but has no points.

8.9.4.26 void Parma_Polyhedra_Library::Polyhedron::add_generators (GenSys & *gs*)

Adds the specified generators without minimizing.

Parameters:

gs The generators that will be added to the current system of generators. This parameter is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument thrown if ∗`this` and `gs` are topology-incompatible or dimension-incompatible, or if ∗`this` is empty and the system of generators `gs` is not empty, but has no points.

8.9.4.27 bool Parma_Polyhedra_Library::Polyhedron::bounds_from_above (const LinExpression & *expr*) const

Returns `true` if and only if `expr` is bounded from above in ∗`this`.

Exceptions:

std::invalid_argument thrown if `expr` and ∗`this` are dimension-incompatible.

8.9.4.28 bool Parma_Polyhedra_Library::Polyhedron::bounds_from_below (const LinExpression & *expr*) const

Returns `true` if and only if `expr` is bounded from below in ∗`this`.

Exceptions:

std::invalid_argument thrown if `expr` and ∗`this` are dimension-incompatible.

8.9.4.29 void Parma_Polyhedra_Library::Polyhedron::swap (Polyhedron & *y*)

Swaps ∗`this` with polyhedron `y`. (∗`this` and `y` can be dimension-incompatible.).

Exceptions:

std::invalid_argument thrown if `x` and `y` are topology-incompatible.

8.9.5 Friends And Related Function Documentation

8.9.5.1 bool Parma_Polyhedra_Library::operator<= (const Polyhedron & *x*, const Polyhedron & *y*) [friend]

Returns `true` if and only if polyhedron `x` is contained in polyhedron `y`.

Exceptions:

std::invalid_argument thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

8.9.5.2 bool operator== (const Polyhedron & *x*, const Polyhedron & *y*) [related]

Returns `true` if and only if `x` and `y` are the same polyhedron.

Exceptions:

std::invalid_argument thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

8.9.5.3 bool operator!= (const Polyhedron & *x*, const Polyhedron & *y*) [related]

Returns `true` if and only if `x` and `y` are different polyhedra.

Exceptions:

std::invalid_argument thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

8.9.5.4 bool operator< (const Polyhedron & *x*, const Polyhedron & *y*) [related]

Returns `true` if and only if `x` is strictly contained in `y`.

Exceptions:

std::invalid_argument thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

8.9.5.5 bool operator> (const Polyhedron & *x*, const Polyhedron & *y*) `[related]`

Returns `true` if and only if `x` strictly contains `y`.

Exceptions:

std::invalid_argument thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

8.9.5.6 bool operator>= (const Polyhedron & *x*, const Polyhedron & *y*) `[related]`

Returns `true` if and only if `x` contains `y`.

Exceptions:

std::invalid_argument thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

8.10 Parma_Polyhedra_Library::Throwable Class Reference

User objects' the PPL can throw.

Public Methods

- virtual void throw_me () const=0

  Throws the user defined exception object.

8.10.1 Detailed Description

User objects' the PPL can throw.

This abstract base class should be instantiated by those users willing to provide a polynomial upper bound to the time spent by any invocation of a library operator.

8.11 Parma_Polyhedra_Library::Variable Class Reference

A dimension of the space.

Public Methods

- Variable (unsigned int i)

  Builds the variable corresponding to the Cartesian axis of index `i`.

- unsigned int id () const

  Returns the index of the Cartesian axis associated to the variable.

Related Functions

(Note that these are not member functions.)

- std::ostream & operator<< (std::ostream &s, const Variable &v)

Output operator.

- bool operator< (const Variable &v, const Variable &w)

    Defines a total ordering on variables.

8.11.1 Detailed Description

A dimension of the space.

An object of the class Variable represents a dimension of the space, that is one of the Cartesian axes. Variables are used as base blocks in order to build more complex linear expressions. Each variable is identified by a non-negative integer, representing the index of the corresponding Cartesian axis (the first axis has index 0).

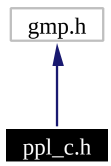
Note that the “meaning” of an object of the class Variable is completely specified by the integer index provided to its constructor: be careful not to be mislead by C++ language variable names. For instance, in the following example the linear expressions `e1` and `e2` are equivalent, since the two variables `x` and `z` denote the same Cartesian axis.

```
Variable x(0);
Variable y(1);
Variable z(0);
LinExpression e1 = x + y;
LinExpression e2 = y + z;
```

9 PPL File Documentation

9.1 ppl_c.h File Reference

Include dependency graph for ppl_c.h:



Typedefs

- typedef ppl_Coefficient_tag ∗ ppl_Coefficient_t

    Opaque pointer to Coefficient .

- typedef ppl_Coefficient_tag const ∗ ppl_const_Coefficient_t

    Opaque pointer to const Coefficient .

- typedef ppl_LinExpression_tag ∗ ppl_LinExpression_t

    Opaque pointer to LinExpression .

- typedef ppl_LinExpression_tag const * ppl_const_LinExpression_t

    Opaque pointer to const LinExpression .

- typedef ppl_Constraint_tag * ppl_Constraint_t

    Opaque pointer to Constraint .

- typedef ppl_Constraint_tag const * ppl_const_Constraint_t

    Opaque pointer to const Constraint .

- typedef ppl_ConSys_tag * ppl_ConSys_t

    Opaque pointer to ConSys .

- typedef ppl_ConSys_tag const * ppl_const_ConSys_t

    Opaque pointer to const ConSys .

- typedef ppl_ConSys__const_iterator_tag * ppl_ConSys__const_iterator_t

    Opaque pointer to ConSys__const_iterator .

- typedef ppl_ConSys__const_iterator_tag const * ppl_const_ConSys__const_iterator_t

    Opaque pointer to const ConSys__const_iterator .

- typedef ppl_Generator_tag * ppl_Generator_t

    Opaque pointer to Generator .

- typedef ppl_Generator_tag const * ppl_const_Generator_t

    Opaque pointer to const Generator .

- typedef ppl_GenSys_tag * ppl_GenSys_t

    Opaque pointer to GenSys .

- typedef ppl_GenSys_tag const * ppl_const_GenSys_t

    Opaque pointer to const GenSys .

- typedef ppl_GenSys__const_iterator_tag * ppl_GenSys__const_iterator_t

    Opaque pointer to GenSys__const_iterator .

- typedef ppl_GenSys__const_iterator_tag const * ppl_const_GenSys__const_iterator_t

    Opaque pointer to const GenSys__const_iterator .

- typedef ppl_Polyhedron_tag * ppl_Polyhedron_t

    Opaque pointer to Polyhedron .

- typedef ppl_Polyhedron_tag const * ppl_const_Polyhedron_t

    Opaque pointer to const Polyhedron .

Enumerations

- enum ppl_enum_error_code { PPL_ERROR_OUT_OF_MEMORY, PPL_ERROR_INVALID_ARGUMENT, PPL_ERROR_INTERNAL_ERROR, PPL_ERROR_UNKNOWN_STANDARD_EXCEPTION, PPL_ERROR_UNEXPECTED_ERROR }
- enum ppl_enum_Constraint_Type { PPL_CONSTRAINT_TYPE_EQUAL, PPL_CONSTRAINT_TYPE_GREATER_THAN_OR_EQUAL, PPL_CONSTRAINT_TYPE_GREATER_THAN, PPL_CONSTRAINT_TYPE_LESS_THAN_OR_EQUAL, PPL_CONSTRAINT_TYPE_LESS_THAN }
- enum ppl_enum_Generator_Type { PPL_GENERATOR_TYPE_LINE, PPL_GENERATOR_TYPE_RAY, PPL_GENERATOR_TYPE_POINT, PPL_GENERATOR_TYPE_CLOSURE_POINT }

Functions

- int ppl_initialize (void)
- int ppl_finalize (void)
- int ppl_set_error_handler (void(*h)(enum ppl_enum_error_code code, const char *description))
- int ppl_new_Coefficient (ppl_Coefficient_t *pc)
- int ppl_new_Coefficient_from_mpz_t (ppl_Coefficient_t *pc, mpz_t z)
- int ppl_new_Coefficient_from_Coefficient (ppl_Coefficient_t *pc, ppl_const_Coefficient_t c)
- int ppl_assign_Coefficient_from_mpz_t (ppl_Coefficient_t dst, mpz_t z)
- int ppl_assign_Coefficient_from_Coefficient (ppl_Coefficient_t dst, ppl_const_Coefficient_t src)
- int ppl_delete_Coefficient (ppl_const_Coefficient_t c)
- int ppl_Coefficient_to_mpz_t (ppl_const_Coefficient_t c, mpz_t z)
- int ppl_Coefficient_OK (ppl_const_Coefficient_t c)
- int ppl_new_LinExpression (ppl_LinExpression_t *ple)
- int ppl_new_LinExpression_with_dimension (ppl_LinExpression_t *ple, unsigned int d)
- int ppl_new_LinExpression_from_LinExpression (ppl_LinExpression_t *ple, ppl_const_LinExpression_t le)
- int ppl_new_LinExpression_from_Constraint (ppl_LinExpression_t *ple, ppl_const_Constraint_t c)
- int ppl_new_LinExpression_from_Generator (ppl_LinExpression_t *ple, ppl_const_Generator_t g)
- int ppl_delete_LinExpression (ppl_const_LinExpression_t le)
- int ppl_assign_LinExpression_from_LinExpression (ppl_LinExpression_t dst, ppl_const_LinExpression_t src)
- int ppl_LinExpression_add_to_coefficient (ppl_LinExpression_t le, unsigned int var, ppl_const_Coefficient_t n)
- int ppl_LinExpression_add_to_inhomogeneous (ppl_LinExpression_t le, ppl_const_Coefficient_t n)
- int ppl_LinExpression_space_dimension (ppl_const_LinExpression_t le)
- int ppl_LinExpression_OK (ppl_const_LinExpression_t le)
- int ppl_new_Constraint (ppl_Constraint_t *pc, ppl_const_LinExpression_t le, enum ppl_enum_Constraint_Type)
- int ppl_new_Constraint_zero_dim_false (ppl_Constraint_t *pc)
- int ppl_new_Constraint_zero_dim_positivity (ppl_Constraint_t *pc)
- int ppl_new_Constraint_from_Constraint (ppl_Constraint_t *pc, ppl_const_Constraint_t c)
- int ppl_delete_Constraint (ppl_const_Constraint_t c)
- int ppl_assign_Constraint_from_Constraint (ppl_Constraint_t dst, ppl_const_Constraint_t src)
- int ppl_Constraint_space_dimension (ppl_const_Constraint_t c)
- int ppl_Constraint_type (ppl_const_Constraint_t c)
- int ppl_Constraint_coefficient (ppl_const_Constraint_t c, int var, ppl_Coefficient_t n)
- int ppl_Constraint_inhomogeneous_term (ppl_const_Constraint_t c, ppl_Coefficient_t n)
- int ppl_Constraint_OK (ppl_const_Constraint_t c)
- int ppl_new_ConSys (ppl_ConSys_t *pcs)

- int ppl_new_ConSys_zero_dim_empty (ppl_ConSys_t *pcs)
- int ppl_new_ConSys_from_Constraint (ppl_ConSys_t *pcs, ppl_const_Constraint_t c)
- int ppl_new_ConSys_from_ConSys (ppl_ConSys_t *pcs, ppl_const_ConSys_t cs)
- int ppl_delete_ConSys (ppl_const_ConSys_t cs)
- int ppl_assign_ConSys_from_ConSys (ppl_ConSys_t dst, ppl_const_ConSys_t src)
- int ppl_ConSys_space_dimension (ppl_const_ConSys_t cs)
- int ppl_ConSys_insert_Constraint (ppl_ConSys_t cs, ppl_const_Constraint_t c)
- int ppl_ConSys_OK (ppl_const_ConSys_t c)
- int ppl_new_ConSys__const_iterator (ppl_ConSys__const_iterator_t *pcit)
- int ppl_new_ConSys__const_iterator_from_ConSys__const_iterator (ppl_ConSys__const_iterator_t *pcit, ppl_const_ConSys__const_iterator_t cit)
- int ppl_delete_ConSys__const_iterator (ppl_const_ConSys__const_iterator_t cit)
- int ppl_assign_ConSys__const_iterator_from_ConSys__const_iterator (ppl_ConSys__const_iterator_t dst, ppl_const_ConSys__const_iterator_t src)
- int ppl_ConSys_begin (ppl_ConSys_t cs, ppl_ConSys__const_iterator_t cit)
- int ppl_ConSys_end (ppl_ConSys_t cs, ppl_ConSys__const_iterator_t cit)
- int ppl_ConSys__const_iterator_dereference (ppl_const_ConSys__const_iterator_t cit, ppl_const_Constraint_t *pc)
- int ppl_ConSys__const_iterator_increment (ppl_ConSys__const_iterator_t cit)
- int ppl_ConSys__const_iterator_equal_test (ppl_const_ConSys__const_iterator_t x, ppl_const_ConSys__const_iterator_t y)
- int ppl_new_Generator (ppl_Generator_t *pg, ppl_const_LinExpression_t le, enum ppl_enum_Generator_Type t, ppl_const_Coefficient_t d)
- int ppl_new_Generator_zero_dim_point (ppl_Generator_t *pg)
- int ppl_new_Generator_zero_dim_closure_point (ppl_Generator_t *pg)
- int ppl_new_Generator_from_Generator (ppl_Generator_t *pg, ppl_const_Generator_t g)
- int ppl_delete_Generator (ppl_const_Generator_t g)
- int ppl_assign_Generator_from_Generator (ppl_Generator_t dst, ppl_const_Generator_t src)
- int ppl_Generator_space_dimension (ppl_const_Generator_t g)
- int ppl_Generator_type (ppl_const_Generator_t g)
- int ppl_Generator_coefficient (ppl_const_Generator_t g, int var, ppl_Coefficient_t n)
- int ppl_Generator_divisor (ppl_const_Generator_t g, ppl_Coefficient_t n)
- int ppl_Generator_OK (ppl_const_Generator_t g)
- int ppl_new_GenSys (ppl_GenSys_t *pgs)
- int ppl_new_GenSys_from_Generator (ppl_GenSys_t *pgs, ppl_const_Generator_t g)
- int ppl_new_GenSys_from_GenSys (ppl_GenSys_t *pgs, ppl_const_GenSys_t gs)
- int ppl_delete_GenSys (ppl_const_GenSys_t gs)
- int ppl_assign_GenSys_from_GenSys (ppl_GenSys_t dst, ppl_const_GenSys_t src)
- int ppl_GenSys_space_dimension (ppl_const_GenSys_t gs)
- int ppl_GenSys_insert_Generator (ppl_GenSys_t gs, ppl_const_Generator_t g)
- int ppl_GenSys_OK (ppl_const_GenSys_t c)
- int ppl_new_GenSys__const_iterator (ppl_GenSys__const_iterator_t *pgit)
- int ppl_new_GenSys__const_iterator_from_GenSys__const_iterator (ppl_GenSys__const_iterator_t *pgit, ppl_const_GenSys__const_iterator_t git)
- int ppl_delete_GenSys__const_iterator (ppl_const_GenSys__const_iterator_t git)
- int ppl_assign_GenSys__const_iterator_from_GenSys__const_iterator (ppl_GenSys__const_iterator_t dst, ppl_const_GenSys__const_iterator_t src)
- int ppl_GenSys_begin (ppl_const_GenSys_t gs, ppl_GenSys__const_iterator_t git)
- int ppl_GenSys_end (ppl_const_GenSys_t gs, ppl_GenSys__const_iterator_t git)
- int ppl_GenSys__const_iterator_dereference (ppl_const_GenSys__const_iterator_t git, ppl_const_Generator_t *pg)

- int ppl_GenSys__const_iterator_increment (ppl_GenSys__const_iterator_t git)
- int ppl_GenSys__const_iterator_equal_test (ppl_const_GenSys__const_iterator_t x, ppl_const_GenSys__const_iterator_t y)
- int ppl_new_C_Polyhedron_from_dimension (ppl_Polyhedron_t *pph, unsigned int d)
- int ppl_new_NNC_Polyhedron_from_dimension (ppl_Polyhedron_t *pph, unsigned int d)
- int ppl_new_C_Polyhedron_empty_from_dimension (ppl_Polyhedron_t *pph, unsigned int d)
- int ppl_new_NNC_Polyhedron_empty_from_dimension (ppl_Polyhedron_t *pph, unsigned int d)
- int ppl_new_C_Polyhedron_from_C_Polyhedron (ppl_Polyhedron_t *pph, ppl_const_Polyhedron_t ph)
- int ppl_new_C_Polyhedron_from_NNC_Polyhedron (ppl_Polyhedron_t *pph, ppl_const_Polyhedron_t ph)
- int ppl_new_NNC_Polyhedron_from_C_Polyhedron (ppl_Polyhedron_t *pph, ppl_const_Polyhedron_t ph)
- int ppl_new_NNC_Polyhedron_from_NNC_Polyhedron (ppl_Polyhedron_t *pph, ppl_const_Polyhedron_t ph)
- int ppl_new_C_Polyhedron_from_ConSys (ppl_Polyhedron_t *pph, ppl_ConSys_t cs)
- int ppl_new_NNC_Polyhedron_from_ConSys (ppl_Polyhedron_t *pph, ppl_ConSys_t cs)
- int ppl_new_C_Polyhedron_from_GenSys (ppl_Polyhedron_t *pph, ppl_GenSys_t gs)
- int ppl_new_NNC_Polyhedron_from_GenSys (ppl_Polyhedron_t *pph, ppl_GenSys_t gs)
- int ppl_new_C_Polyhedron_from_bounding_box (ppl_Polyhedron_t *pph, unsigned int(*space_dimension)(void), int(*is_empty)(void), int(*get_lower_bound)(unsigned int k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d), int(*get_upper_bound)(unsigned int k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d))
- int ppl_new_NNC_Polyhedron_from_bounding_box (ppl_Polyhedron_t *pph, unsigned int(*space_dimension)(void), int(*is_empty)(void), int(*get_lower_bound)(unsigned int k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d), int(*get_upper_bound)(unsigned int k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d))
- int ppl_delete_Polyhedron (ppl_const_Polyhedron_t ph)
- int ppl_assign_C_Polyhedron_from_C_Polyhedron (ppl_Polyhedron_t dst, ppl_const_Polyhedron_t src)
- int ppl_assign_NNC_Polyhedron_from_NNC_Polyhedron (ppl_Polyhedron_t dst, ppl_const_Polyhedron_t src)
- int ppl_Polyhedron_space_dimension (ppl_const_Polyhedron_t ph)
- int ppl_Polyhedron_intersection_assign (ppl_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_intersection_assign_and_minimize (ppl_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_poly_hull_assign (ppl_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_poly_hull_assign_and_minimize (ppl_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_poly_difference_assign (ppl_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_poly_difference_assign_and_minimize (ppl_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_H79_widening_assign (ppl_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_limited_H79_widening_assign (ppl_Polyhedron_t x, ppl_const_Polyhedron_t y, ppl_ConSys_t cs)
- int ppl_Polyhedron_constraints (ppl_const_Polyhedron_t ph, ppl_const_ConSys_t *pcs)
- int ppl_Polyhedron_minimized_constraints (ppl_const_Polyhedron_t ph, ppl_const_ConSys_t *pcs)
- int ppl_Polyhedron_generators (ppl_const_Polyhedron_t ph, ppl_const_GenSys_t *pgs)
- int ppl_Polyhedron_minimized_generators (ppl_const_Polyhedron_t ph, ppl_const_GenSys_t *pgs)
- int ppl_Polyhedron_add_constraint (ppl_Polyhedron_t ph, ppl_const_Constraint_t c)
- int ppl_Polyhedron_add_generator (ppl_Polyhedron_t ph, ppl_const_Generator_t g)
- int ppl_Polyhedron_add_constraints (ppl_Polyhedron_t ph, ppl_ConSys_t cs)
- int ppl_Polyhedron_add_constraints_and_minimize (ppl_Polyhedron_t ph, ppl_ConSys_t cs)

- int ppl_Polyhedron_add_generators (ppl_Polyhedron_t ph, ppl_GenSys_t gs)
- int ppl_Polyhedron_add_generators_and_minimize (ppl_Polyhedron_t ph, ppl_GenSys_t gs)
- int ppl_Polyhedron_add_dimensions_and_embed (ppl_Polyhedron_t ph, unsigned int d)
- int ppl_Polyhedron_add_dimensions_and_project (ppl_Polyhedron_t ph, unsigned int d)
- int ppl_Polyhedron_remove_dimensions (ppl_Polyhedron_t ph, unsigned int ds[ ], unsigned int n)
- int ppl_Polyhedron_remove_higher_dimensions (ppl_Polyhedron_t ph, unsigned int d)
- int ppl_Polyhedron_add_dimensions_and_constraints (ppl_Polyhedron_t ph, ppl_ConSys_t cs)
- int ppl_Polyhedron_affine_image (ppl_Polyhedron_t ph, unsigned int var, ppl_const_LinExpression_t le, ppl_const_Coefficient_t d)
- int ppl_Polyhedron_affine_preimage (ppl_Polyhedron_t ph, unsigned int var, ppl_const_LinExpression_t le, ppl_const_Coefficient_t d)
- int ppl_Polyhedron_shrink_bounding_box (ppl_const_Polyhedron_t ph, void(*set_empty)(void), void(*raise_lower_bound)(unsigned int k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d), void(*lower_upper_bound)(unsigned int k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d))
- int ppl_Polyhedron_relation_with_Constraint (ppl_const_Polyhedron_t ph, ppl_const_Constraint_t c)
- int ppl_Polyhedron_relation_with_Generator (ppl_const_Polyhedron_t ph, ppl_const_Generator_t g)
- int ppl_Polyhedron_check_empty (ppl_const_Polyhedron_t ph)
- int ppl_Polyhedron_check_universe (ppl_const_Polyhedron_t ph)
- int ppl_Polyhedron_is_bounded (ppl_const_Polyhedron_t ph)
- int ppl_Polyhedron_bounds_from_above (ppl_const_Polyhedron_t ph, ppl_const_LinExpression_t le)
- int ppl_Polyhedron_bounds_from_below (ppl_const_Polyhedron_t ph, ppl_const_LinExpression_t le)
- int ppl_Polyhedron_is_topologically_closed (ppl_const_Polyhedron_t ph)
- int ppl_Polyhedron_topological_closure_assign (ppl_Polyhedron_t ph)
- int ppl_Polyhedron_contains_Polyhedron (ppl_const_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_strictly_contains_Polyhedron (ppl_const_Polyhedron_t x, ppl_const_Polyhedron_t y)
- int ppl_Polyhedron_OK (ppl_const_Polyhedron_t ph)

Variables

- unsigned int PPL_POLY_CON_RELATION_IS_DISJOINT
- unsigned int PPL_POLY_CON_RELATION_STRICTLY_INTERSECTS
- unsigned int PPL_POLY_CON_RELATION_IS_INCLUDED
- unsigned int PPL_POLY_CON_RELATION_SATURATES
- unsigned int PPL_POLY_GEN_RELATION_SUBSUMES

9.1.1 Detailed Description

This file implements the C interface. Detailed description with examples to be written.

9.1.2 Define Documentation

9.1.2.1 #define PPL_TYPE_DECLARATION(Type)

Value:

```
/*! \brief Opaque pointer to Type. */ \
typedef struct ppl_ ## Type ## _tag* ppl_ ## Type ## _t; \
/*! \brief Opaque pointer to const Type. */ \
typedef struct ppl_ ## Type ## _tag const* ppl_const_ ## Type ## _t
```

9.1.3 Enumeration Type Documentation

9.1.3.1 enum ppl_enum_error_code

Defines the error code that any function can return.

Enumeration values:

PPL_ERROR_OUT_OF_MEMORY The virtual memory available to the process has been exhausted.

PPL_ERROR_INVALID_ARGUMENT A function has been invoked with an invalid argument.

PPL_ERROR_INTERNAL_ERROR An internal error that was diagnosed by the PPL itself. This indicates a bug in the PPL.

PPL_ERROR_UNKNOWN_STANDARD_EXCEPTION A standard exception has been raised by the C++ run-time environment. This indicates a bug in the PPL.

PPL_ERROR_UNEXPECTED_ERROR A totally unknown, totally unexpected error happened. This indicates a bug in the PPL.

9.1.3.2 enum ppl_enum_Constraint_Type

Describes the relations represented by a constraint.

Enumeration values:

PPL_CONSTRAINT_TYPE_EQUAL The constraint is of the form $e = 0$.

PPL_CONSTRAINT_TYPE_GREATER_THAN_OR_EQUAL The constraint is of the form $e \geq 0$.

PPL_CONSTRAINT_TYPE_GREATER_THAN The constraint is of the form $e > 0$.

PPL_CONSTRAINT_TYPE_LESS_THAN_OR_EQUAL The constraint is of the form $e \leq 0$.

PPL_CONSTRAINT_TYPE_LESS_THAN The constraint is of the form $e < 0$.

9.1.3.3 enum ppl_enum_Generator_Type

Describes the different kinds of generators.

Enumeration values:

PPL_GENERATOR_TYPE_LINE The generator is a line.

PPL_GENERATOR_TYPE_RAY The generator is a ray.

PPL_GENERATOR_TYPE_POINT The generator is a point.

PPL_GENERATOR_TYPE_CLOSURE_POINT The generator is a closure point.

9.1.4 Function Documentation

9.1.4.1 int ppl_initialize (void)

Initializes the Parma Polyhedra Library. This function must be called before any other function.

9.1.4.2 int ppl_finalize (void)

Finalizes the Parma Polyhedra Library. This function must be called after any other function.

9.1.4.3 int ppl_set_error_handler (void(∗ *h*)(enum ppl_enum_error_code code, const char ∗description))

Installs the user-defined error handler pointed by `h`. The error handler takes an error code and a textual description that gives further information about the actual error. The C string containing the textual description is read-only and its existence it not guaranteed after the handler has returned.

9.1.4.4 int ppl_new_Coefficient (ppl_Coefficient_t ∗ *pc*)

Creates a new coefficent with value 0 and writes an handle for the newly created coefficient at address `pc`.

9.1.4.5 int ppl_new_Coefficient_from_mpz_t (ppl_Coefficient_t ∗ *pc*, mpz_t *z*)

Creates a new coefficent with the value given by the GMP integer `z` and writes an handle for the newly created coefficient at address `pc`.

9.1.4.6 int ppl_new_Coefficient_from_Coefficient (ppl_Coefficient_t ∗ *pc*, ppl_const_Coefficient_t *c*)

Builds a coefficient that is a copy of `c`; writes an handle for the newly created coefficient at address `pc`.

9.1.4.7 int ppl_assign_Coefficient_from_mpz_t (ppl_Coefficient_t *dst*, mpz_t *z*)

Assign to `dst` the value given by the GMP integer `z`.

9.1.4.8 int ppl_assign_Coefficient_from_Coefficient (ppl_Coefficient_t *dst*, ppl_const_Coefficient_t *src*)

Assigns a copy of the linear expression `src` to `dst`.

9.1.4.9 int ppl_delete_Coefficient (ppl_const_Coefficient_t *c*)

Invalidates the handle `c`: this makes sure the corresponding resources will eventually be released.

9.1.4.10 int ppl_Coefficient_to_mpz_t (ppl_const_Coefficient_t *c*, mpz_t *z*)

Sets the value of the GMP integer `z` to the value of `c`.

9.1.4.11 int ppl_Coefficient_OK (ppl_const_Coefficient_t *c*)

Returns a positive integer if `c` is well formed, i.e., if it satisfies all its implementation variant; returns 0 and perhaps make some noise if `c` is broken. Useful for debugging purposes.

9.1.4.12 int ppl_new_LinExpression (ppl_LinExpression_t ∗ *ple*)

Creates a new linear expression corresponding to the constant 0 in a zero-dimensional space; writes an handle for the new linear expression at address `ple`.

9.1.4.13 int ppl_new_LinExpression_with_dimension (ppl_LinExpression_t ∗ *ple*, unsigned int *d*)

Creates a new linear expression corresponding to the constant 0 in a `d`-dimensional space; writes an handle for the new linear expression at address `ple`.

9.1.4.14 int ppl_new_LinExpression_from_LinExpression (ppl_LinExpression_t * ple, ppl_const_LinExpression_t le)

Builds a linear expression that is a copy of `le`; writes an handle for the newly created linear expression at address `ple`.

9.1.4.15 int ppl_new_LinExpression_from_Constraint (ppl_LinExpression_t * ple, ppl_const_Constraint_t c)

Builds a linear expression corresponding to constraint `c`; writes an handle for the newly created linear expression at address `ple`.

9.1.4.16 int ppl_new_LinExpression_from_Generator (ppl_LinExpression_t * ple, ppl_const_Generator_t g)

Builds a linear expression corresponding to generator `g`; writes an handle for the newly created linear expression at address `ple`.

9.1.4.17 int ppl_delete_LinExpression (ppl_const_LinExpression_t le)

Invalidates the handle `le`: this makes sure the corresponding resources will eventually be released.

9.1.4.18 int ppl_assign_LinExpression_from_LinExpression (ppl_LinExpression_t dst, ppl_const_LinExpression_t src)

Assigns a copy of the linear expression `src` to `dst`.

9.1.4.19 int ppl_LinExpression_add_to_coefficient (ppl_LinExpression_t le, unsigned int var, ppl_const_Coefficient_t n)

Adds `n` to the coefficient of variable `var` in the linear expression `le`. The space dimension is set to be the maximum between `var` + 1 and the old space dimension.

9.1.4.20 int ppl_LinExpression_add_to_inhomogeneous (ppl_LinExpression_t le, ppl_const_Coefficient_t n)

Adds `n` to the inhomogeneous term of the linear expression `le`.

9.1.4.21 int ppl_LinExpression_space_dimension (ppl_const_LinExpression_t le)

Returns the space dimension of `le`.

9.1.4.22 int ppl_LinExpression_OK (ppl_const_LinExpression_t le)

Returns a positive integer if `le` is well formed, i.e., if it satisfies all its implementation variant; returns 0 and perhaps make some noise if `le` is broken. Useful for debugging purposes.

9.1.4.23 int ppl_new_Constraint (ppl_Constraint_t * pc, ppl_const_LinExpression_t le, enum ppl_enum_Constraint_Type)

Creates the new constraint '`le rel` 0' and writes an handle for it at address `pc`. The space dimension of the new constraint is equal to the space dimension of `le`.

9.1.4.24 int ppl_new_Constraint_zero_dim_false (ppl_Constraint_t * *pc*)

Creates the unsatisfiable (zero-dimension space) constraint $0 = 1$ and writes an handle for it at address `pc`.

9.1.4.25 int ppl_new_Constraint_zero_dim_positivity (ppl_Constraint_t * *pc*)

Creates the true (zero-dimension space) constraint $0 \leq 1$, also known as *positivity constraint*. An handle for the newly created constraint is written at address `pc`.

9.1.4.26 int ppl_new_Constraint_from_Constraint (ppl_Constraint_t * *pc*, ppl_const_Constraint_t *c*)

Builds a constraint that is a copy of `c`; writes an handle for the newly created constraint at address `pc`.

9.1.4.27 int ppl_delete_Constraint (ppl_const_Constraint_t *c*)

Invalidates the handle `c`: this makes sure the corresponding resources will eventually be released.

9.1.4.28 int ppl_assign_Constraint_from_Constraint (ppl_Constraint_t *dst*, ppl_const_Constraint_t *src*)

Assigns a copy of the constraint `src` to `dst`.

9.1.4.29 int ppl_Constraint_space_dimension (ppl_const_Constraint_t *c*)

Returns the space dimension of `c`.

9.1.4.30 int ppl_Constraint_type (ppl_const_Constraint_t *c*)

Returns the type of constraint `c`.

9.1.4.31 int ppl_Constraint_coefficient (ppl_const_Constraint_t *c*, int *var*, ppl_Coefficient_t *n*)

Copies into `n` the coefficient of variable `var` in constraint `c`.

9.1.4.32 int ppl_Constraint_inhomogeneous_term (ppl_const_Constraint_t *c*, ppl_Coefficient_t *n*)

Copies into `n` the inhomogeneous term of constraint `c`.

9.1.4.33 int ppl_Constraint_OK (ppl_const_Constraint_t *c*)

Returns a positive integer if `c` is well formed, i.e., if it satisfies all its implementation variant; returns 0 and perhaps make some noise if `c` is broken. Useful for debugging purposes.

9.1.4.34 int ppl_new_ConSys (ppl_ConSys_t * *pcs*)

Builds an empty system of constraints and writes an handle to it at address `pcs`.

9.1.4.35 int ppl_new_ConSys_zero_dim_empty (ppl_ConSys_t * *pcs*)

Builds a zero-dimensional, unsatisfiable constraint system and writes an handle to it at address `pcs`.

9.1.4.36 int ppl_new_ConSys_from_Constraint (ppl_ConSys_t ∗ *pcs*, ppl_const_Constraint_t *c*)

Builds the singleton constraint system containing only a copy of constraint `c`; writes an handle for the newly created system at address `pcs`.

9.1.4.37 int ppl_new_ConSys_from_ConSys (ppl_ConSys_t ∗ *pcs*, ppl_const_ConSys_t *cs*)

Builds a constraint system that is a copy of `cs`; writes an handle for the newly created system at address `pcs`.

9.1.4.38 int ppl_delete_ConSys (ppl_const_ConSys_t *cs*)

Invalidates the handle `cs`: this makes sure the corresponding resources will eventually be released.

9.1.4.39 int ppl_assign_ConSys_from_ConSys (ppl_ConSys_t *dst*, ppl_const_ConSys_t *src*)

Assigns a copy of the constraint system `src` to `dst`.

9.1.4.40 int ppl_ConSys_space_dimension (ppl_const_ConSys_t *cs*)

Returns the dimension of the vector space enclosing ∗`this`.

9.1.4.41 int ppl_ConSys_insert_Constraint (ppl_ConSys_t *cs*, ppl_const_Constraint_t *c*)

Inserts a copy of the constraint `c` into ∗`this`; the space dimension is increased, if necessary.

9.1.4.42 int ppl_ConSys_OK (ppl_const_ConSys_t *c*)

Returns a positive integer if `cs` is well formed, i.e., if it satisfies all its implementation variant; returns 0 and perhaps make some noise if `cs` is broken. Useful for debugging purposes.

9.1.4.43 int ppl_new_ConSys__const_iterator (ppl_ConSys__const_iterator_t ∗ *pcit*)

Builds a new 'const iterator' and writes an handle to it at address `pcit`.

9.1.4.44 int ppl_new_ConSys__const_iterator_from_ConSys__const_iterator (ppl_ConSys__const_iterator_t ∗ *pcit*, ppl_const_ConSys__const_iterator_t *cit*)

Builds a const iterator system that is a copy of `cit`; writes an handle for the newly created const iterator at address `pcit`.

9.1.4.45 int ppl_delete_ConSys__const_iterator (ppl_const_ConSys__const_iterator_t *cit*)

Invalidates the handle `cit`: this makes sure the corresponding resources will eventually be released.

9.1.4.46 int ppl_assign_ConSys__const_iterator_from_ConSys__const_iterator (ppl_ConSys__const_iterator_t *dst*, ppl_const_ConSys__const_iterator_t *src*)

Assigns a copy of the const iterator `src` to `dst`.

9.1.4.47 int ppl_ConSys_begin (ppl_ConSys_t *cs*, ppl_ConSys__const_iterator_t *cit*)

Assigns to `cit` a const iterator "pointing" to the beginning of the constraint system `cs`.

9.1.4.48 int ppl_ConSys_end (ppl_ConSys_t *cs*, ppl_ConSys__const_iterator_t *cit*)

Assigns to `cit` a const iterator "pointing" past the end of the constraint system `cs`.

9.1.4.49 int ppl_ConSys__const_iterator_dereference (ppl_const_ConSys__const_iterator_t *cit*, ppl_const_Constraint_t * *pc*)

Dereference `cit` writing a const handle to the resulting constraint at address `pc`.

9.1.4.50 int ppl_ConSys__const_iterator_increment (ppl_ConSys__const_iterator_t *cit*)

Increment `cit` so that it "points" to the next constraint.

9.1.4.51 int ppl_ConSys__const_iterator_equal_test (ppl_const_ConSys__const_iterator_t *x*, ppl_const_ConSys__const_iterator_t *y*)

Returns a positive integer if the iterators corresponding to `x` and `y` are equal; return 0 if they are different.

9.1.4.52 int ppl_new_Generator (ppl_Generator_t * *pg*, ppl_const_LinExpression_t *le*, enum ppl_enum_Generator_Type *t*, ppl_const_Coefficient_t *d*)

Creates a new generator of direction `le` and type `t`. If the generator to be created is a point or a closure point, the divisor `d` is applied to `le`. For other types of generators `d` is simply disregarded. An handle for the new generator is written at address `pg`. The space dimension of the new generator is equal to the space dimension of `le`.

9.1.4.53 int ppl_new_Generator_zero_dim_point (ppl_Generator_t * *pg*)

Creates the point that is the origin of the zero-dimensional space $\mathbb{R}^0$. Writes an handle for the new generator at address `pg`.

9.1.4.54 int ppl_new_Generator_zero_dim_closure_point (ppl_Generator_t * *pg*)

Creates, as a closure point, the point that is the origin of the zero-dimensional space $\mathbb{R}^0$. Writes an handle for the new generator at address `pg`.

9.1.4.55 int ppl_new_Generator_from_Generator (ppl_Generator_t * *pg*, ppl_const_Generator_t *g*)

Builds a generator that is a copy of `g`; writes an handle for the newly created generator at address `pg`.

9.1.4.56 int ppl_delete_Generator (ppl_const_Generator_t *g*)

Invalidates the handle `g`: this makes sure the corresponding resources will eventually be released.

9.1.4.57 int ppl_assign_Generator_from_Generator (ppl_Generator_t *dst*, ppl_const_Generator_t *src*)

Assigns a copy of the generator `src` to `dst`.

9.1.4.58 int ppl_Generator_space_dimension (ppl_const_Generator_t *g*)

Returns the space dimension of `g`.

9.1.4.59 int ppl_Generator_type (ppl_const_Generator_t *g*)

Returns the type of generator g.

9.1.4.60 int ppl_Generator_coefficient (ppl_const_Generator_t *g*, int *var*, ppl_Coefficient_t *n*)

Copies into n the coefficient of variable var in generator g.

9.1.4.61 int ppl_Generator_divisor (ppl_const_Generator_t *g*, ppl_Coefficient_t *n*)

If g is a point or a closure point assigns its divisor to n.

9.1.4.62 int ppl_Generator_OK (ppl_const_Generator_t *g*)

Returns a positive integer if g is well formed, i.e., if it satisfies all its implementation variant; returns 0 and perhaps make some noise if g is broken. Useful for debugging purposes.

9.1.4.63 int ppl_new_GenSys (ppl_GenSys_t ∗ *pgs*)

Builds an empty system of generators and writes an handle to it at address pgs.

9.1.4.64 int ppl_new_GenSys_from_Generator (ppl_GenSys_t ∗ *pgs*, ppl_const_Generator_t *g*)

Builds the singleton generator system containing only a copy of generator g; writes an handle for the newly created system at address pgs.

9.1.4.65 int ppl_new_GenSys_from_GenSys (ppl_GenSys_t ∗ *pgs*, ppl_const_GenSys_t *gs*)

Builds a generator system that is a copy of gs; writes an handle for the newly created system at address pgs.

9.1.4.66 int ppl_delete_GenSys (ppl_const_GenSys_t *gs*)

Invalidates the handle gs: this makes sure the corresponding resources will eventually be released.

9.1.4.67 int ppl_assign_GenSys_from_GenSys (ppl_GenSys_t *dst*, ppl_const_GenSys_t *src*)

Assigns a copy of the generator system src to dst.

9.1.4.68 int ppl_GenSys_space_dimension (ppl_const_GenSys_t *gs*)

Returns the dimension of the vector space enclosing ∗this.

9.1.4.69 int ppl_GenSys_insert_Generator (ppl_GenSys_t *gs*, ppl_const_Generator_t *g*)

Inserts a copy of the generator g into ∗this; the space dimension is increased, if necessary.

9.1.4.70 int ppl_GenSys_OK (ppl_const_GenSys_t *c*)

Returns a positive integer if gs is well formed, i.e., if it satisfies all its implementation variant; returns 0 and perhaps make some noise if gs is broken. Useful for debugging purposes.

9.1.4.71 int ppl_new_GenSys__const_iterator (ppl_GenSys__const_iterator_t * *pgit*)

Builds a new 'const iterator' and writes an handle to it at address `pgit`.

9.1.4.72 int ppl_new_GenSys__const_iterator_from_GenSys__const_iterator (ppl_GenSys__const_iterator_t * *pgit*, ppl_const_GenSys__const_iterator_t *git*)

Builds a const iterator system that is a copy of `git`; writes an handle for the newly created const iterator at address `pgit`.

9.1.4.73 int ppl_delete_GenSys__const_iterator (ppl_const_GenSys__const_iterator_t *git*)

Invalidates the handle `git`: this makes sure the corresponding resources will eventually be released.

9.1.4.74 int ppl_assign_GenSys__const_iterator_from_GenSys__const_iterator (ppl_GenSys__const_iterator_t *dst*, ppl_const_GenSys__const_iterator_t *src*)

Assigns a copy of the const iterator `src` to `dst`.

9.1.4.75 int ppl_GenSys_begin (ppl_const_GenSys_t *gs*, ppl_GenSys__const_iterator_t *git*)

Assigns to `git` a const iterator "pointing" to the beginning of the generator system `gs`.

9.1.4.76 int ppl_GenSys_end (ppl_const_GenSys_t *gs*, ppl_GenSys__const_iterator_t *git*)

Assigns to `git` a const iterator "pointing" past the end of the generator system `gs`.

9.1.4.77 int ppl_GenSys__const_iterator_dereference (ppl_const_GenSys__const_iterator_t *git*, ppl_const_Generator_t * *pg*)

Dereference `git` writing a const handle to the resulting generator at address `pg`.

9.1.4.78 int ppl_GenSys__const_iterator_increment (ppl_GenSys__const_iterator_t *git*)

Increment `git` so that it "points" to the next generator.

9.1.4.79 int ppl_GenSys__const_iterator_equal_test (ppl_const_GenSys__const_iterator_t *x*, ppl_const_GenSys__const_iterator_t *y*)

Return a positive integer if the iterators corresponding to `x` and `y` are equal; return 0 if they are different.

9.1.4.80 int ppl_new_C_Polyhedron_from_dimension (ppl_Polyhedron_t * *pph*, unsigned int *d*)

Builds an universe closed polyhedron of dimension `d` and writes an handle to it at address `pph`.

9.1.4.81 int ppl_new_NNC_Polyhedron_from_dimension (ppl_Polyhedron_t * *pph*, unsigned int *d*)

Builds an universe NNC polyhedron of dimension `d` and writes an handle to it at address `pph`.

9.1.4.82 int ppl_new_C_Polyhedron_empty_from_dimension (ppl_Polyhedron_t * *pph*, unsigned int *d*)

Builds an empty closed polyhedron of dimension `d` and writes an handle to it at address `pph`.

9.1.4.83 int ppl_new_NNC_Polyhedron_empty_from_dimension (ppl_Polyhedron_t ∗ *pph*, unsigned int *d*)

Builds an empty NNC polyhedron of dimension `d` and writes an handle to it at address `pph`.

9.1.4.84 int ppl_new_C_Polyhedron_from_C_Polyhedron (ppl_Polyhedron_t ∗ *pph*, ppl_const_Polyhedron_t *ph*)

Builds a closed polyhedron that is a copy of `ph`; writes an handle for the newly created polyhedron at address `pph`.

9.1.4.85 int ppl_new_C_Polyhedron_from_NNC_Polyhedron (ppl_Polyhedron_t ∗ *pph*, ppl_const_Polyhedron_t *ph*)

Builds a closed polyhedron that is a copy of of the NNC polyhedron `ph`; writes an handle for the newly created polyhedron at address `pph`.

9.1.4.86 int ppl_new_NNC_Polyhedron_from_C_Polyhedron (ppl_Polyhedron_t ∗ *pph*, ppl_const_Polyhedron_t *ph*)

Builds an NNC polyhedron that is a copy of of the closed polyhedron `ph`; writes an handle for the newly created polyhedron at address `pph`.

9.1.4.87 int ppl_new_NNC_Polyhedron_from_NNC_Polyhedron (ppl_Polyhedron_t ∗ *pph*, ppl_const_Polyhedron_t *ph*)

Builds an NNC polyhedron that is a copy of `ph`; writes an handle for the newly created polyhedron at address `pph`.

9.1.4.88 int ppl_new_C_Polyhedron_from_ConSys (ppl_Polyhedron_t ∗ *pph*, ppl_ConSys_t *cs*)

Builds a new closed polyhedron recycling the system of constraints `cs` and writes an handle for the newly created polyhedron at address `pph`. Since `cs` will be *the* system of constraints of the new polyhedron, the space dimension is also inherited.

Warning:

This function modifies the constraint system referenced by `cs`: upon return, no assumption can be made on its value.

9.1.4.89 int ppl_new_NNC_Polyhedron_from_ConSys (ppl_Polyhedron_t ∗ *pph*, ppl_ConSys_t *cs*)

Builds a new NNC polyhedron recycling the system of constraints `cs` and writes an handle for the newly created polyhedron at address `pph`. Since `cs` will be *the* system of constraints of the new polyhedron, the space dimension is also inherited.

Warning:

This function modifies the constraint system referenced by `cs`: upon return, no assumption can be made on its value.

9.1.4.90 int ppl_new_C_Polyhedron_from_GenSys (ppl_Polyhedron_t * *pph*, ppl_GenSys_t *gs*)

Builds a new closed polyhedron recycling the system of generators `gs` and writes an handle for the newly created polyhedron at address `pph`. Since `cs` will be *the* system of generators of the new polyhedron, the space dimension is also inherited.

Warning:
This function modifies the generator system referenced by `gs`: upon return, no assumption can be made on its value.

9.1.4.91 int ppl_new_NNC_Polyhedron_from_GenSys (ppl_Polyhedron_t * *pph*, ppl_GenSys_t *gs*)

Builds a new NNC polyhedron recycling the system of generators `gs` and writes an handle for the newly created polyhedron at address `pph`. Since `cs` will be *the* system of generators of the new polyhedron, the space dimension is also inherited.

Warning:
This function modifies the generator system referenced by `gs`: upon return, no assumption can be made on its value.

9.1.4.92 int ppl_new_C_Polyhedron_from_bounding_box (ppl_Polyhedron_t * *pph*, unsigned int(* *space_dimension*)(void), int(* *is_empty*)(void), int(* *get_lower_bound*)(unsigned int k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d), int(* *get_upper_bound*)(unsigned int k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d))

Builds a new C polyhedron corresponding to an interval-based bounding box, writing a handle for the newly created polyhedron at address `pph`. If an interval of the bounding box is provided with any finite but open bound, then the polyhedron is not built and the value `PPL_ERROR_INVALID_ARGUMENT` is returned. The bounding box is accessed by using the following functions, passed as arguments:

```
unsigned int space_dimension()
```

returns the dimension of the vector space enclosing the polyhedron represented by the bounding box.

```
int is_empty()
```

returns 0 if and only if the bounding box describes a non-empty set. The function `is_empty()` will always be called before the other functions. However, if `is_empty()` does not return 0, none of the functions below will be called.

```
int get_lower_bound(unsigned int k, int closed,
                    ppl_Coefficient_t n, ppl_Coefficient_t d)
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from below, simply return 0. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to 0 if the lower boundary of I is open and is set to a value different from zero otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the greatest lower bound of I. The fraction n/d is in canonical form if and only if n and d have no common factors and d is positive, $0/1$ being the unique representation for zero.

```
int get_upper_bound(unsigned int k, int closed,
                    ppl_Coefficient_t n, ppl_Coefficient_t d)
```

Let I the interval corresponding to the k-th dimension. If I is not bounded from above, simply return 0. Otherwise, set closed, n and d as follows: closed is set to 0 if the upper boundary of I is open and is set to a value different from 0 otherwise; n and d are assigned the integers n and d such that the canonical fraction n/d corresponds to the least upper bound of I.

9.1.4.93 int ppl_new_NNC_Polyhedron_from_bounding_box (ppl_Polyhedron_t ∗ *pph*, unsigned int(∗ *space_dimension*)(void), int(∗ *is_empty*)(void), int(∗ *get_lower_bound*)(unsigned int k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d), int(∗ *get_upper_bound*)(unsigned int k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d))

Builds a new C polyhedron corresponding to an interval-based bounding box, writing a handle for the newly created polyhedron at address pph. The bounding box is accessed by using the following functions, passed as arguments:

```
unsigned int space_dimension()
```

returns the dimension of the vector space enclosing the polyhedron represented by the bounding box.

```
int is_empty()
```

returns 0 if and only if the bounding box describes a non-empty set. The function is_empty() will always be called before the other functions. However, if is_empty() does not return 0, none of the functions below will be called.

```
int get_lower_bound(unsigned int k, int closed,
                    ppl_Coefficient_t n, ppl_Coefficient_t d)
```

Let I the interval corresponding to the k-th dimension. If I is not bounded from below, simply return 0. Otherwise, set closed, n and d as follows: closed is set to 0 if the lower boundary of I is open and is set to a value different from zero otherwise; n and d are assigned the integers n and d such that the canonical fraction n/d corresponds to the greatest lower bound of I. The fraction n/d is in canonical form if and only if n and d have no common factors and d is positive, $0/1$ being the unique representation for zero.

```
int get_upper_bound(unsigned int k, int closed,
                    ppl_Coefficient_t n, ppl_Coefficient_t d)
```

Let I the interval corresponding to the k-th dimension. If I is not bounded from above, simply return 0. Otherwise, set closed, n and d as follows: closed is set to 0 if the upper boundary of I is open and is set to a value different from 0 otherwise; n and d are assigned the integers n and d such that the canonical fraction n/d corresponds to the least upper bound of I.

9.1.4.94 int ppl_delete_Polyhedron (ppl_const_Polyhedron_t *ph*)

Invalidates the handle ph: this makes sure the corresponding resources will eventually be released.

9.1.4.95 int ppl_assign_C_Polyhedron_from_C_Polyhedron (ppl_Polyhedron_t *dst*, ppl_const_Polyhedron_t *src*)

Assigns a copy of the closed polyhedron src to the closed polyhedron dst.

9.1.4.96 int ppl_assign_NNC_Polyhedron_from_NNC_Polyhedron (ppl_Polyhedron_t *dst*, ppl_const_Polyhedron_t *src*)

Assigns a copy of the NNC polyhedron `src` to the NNC polyhedron `dst`.

9.1.4.97 int ppl_Polyhedron_space_dimension (ppl_const_Polyhedron_t *ph*)

Returns the dimension of the vector space enclosing `ph`.

9.1.4.98 int ppl_Polyhedron_intersection_assign (ppl_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Intersects `x` with polyhedron `y` and assigns the result `x`.

9.1.4.99 int ppl_Polyhedron_intersection_assign_and_minimize (ppl_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Intersects `x` with polyhedron `y` and assigns the result `x`. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return, `x` is also guaranteed to be minimized.

9.1.4.100 int ppl_Polyhedron_poly_hull_assign (ppl_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Assigns to `x` the poly-hull of the set-theoretic union of `x` and `y`.

9.1.4.101 int ppl_Polyhedron_poly_hull_assign_and_minimize (ppl_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Assigns to `x` the poly-hull of the set-theoretic union of `x` and `y`. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return, `x` is also guaranteed to be minimized.

9.1.4.102 int ppl_Polyhedron_poly_difference_assign (ppl_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Assigns to `x` the poly-hull of the set-theoretic difference of `x` and `y`.

9.1.4.103 int ppl_Polyhedron_poly_difference_assign_and_minimize (ppl_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Assigns to `x` the poly-hull of the set-theoretic difference of `x` and `y`. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return, `x` is also guaranteed to be minimized.

9.1.4.104 int ppl_Polyhedron_H79_widening_assign (ppl_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

If the polyhedron `y` is contained in (or equal to) the polyhedron `x`, assigns to `x` the H79-widening of `x` and `y`.

9.1.4.105 int ppl_Polyhedron_limited_H79_widening_assign (ppl_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*, ppl_ConSys_t *cs*)

If the polyhedron y is contained in (or equal to) the polyhedron x, assigns to x the H79-widening of x and y intersected with the constraint system cs.

Warning:

This function modifies the constraint system referenced by cs: upon return, no assumption can be made on its value.

9.1.4.106 int ppl_Polyhedron_constraints (ppl_const_Polyhedron_t *ph*, ppl_const_ConSys_t ∗ *pcs*)

Writes a const handle to the constraint system defining the polyhedron ph at address pcs.

9.1.4.107 int ppl_Polyhedron_minimized_constraints (ppl_const_Polyhedron_t *ph*, ppl_const_ConSys_t ∗ *pcs*)

Writes a const handle to the minimized constraint system defining the polyhedron ph at address pcs.

9.1.4.108 int ppl_Polyhedron_generators (ppl_const_Polyhedron_t *ph*, ppl_const_GenSys_t ∗ *pgs*)

Writes a const handle to the generator system defining the polyhedron ph at address pgs.

9.1.4.109 int ppl_Polyhedron_minimized_generators (ppl_const_Polyhedron_t *ph*, ppl_const_GenSys_t ∗ *pgs*)

Writes a const handle to the minimized generator system defining the polyhedron ph at address pgs.

9.1.4.110 int ppl_Polyhedron_add_constraint (ppl_Polyhedron_t *ph*, ppl_const_Constraint_t *c*)

Adds a copy of the constraint c to the system of constraints of ph.

9.1.4.111 int ppl_Polyhedron_add_generator (ppl_Polyhedron_t *ph*, ppl_const_Generator_t *g*)

Adds a copy of the generator g to the system of generatorss of ph.

9.1.4.112 int ppl_Polyhedron_add_constraints (ppl_Polyhedron_t *ph*, ppl_ConSys_t *cs*)

Adds the system of constraints cs to the system of constraints of ph.

Warning:

This function modifies the constraint system referenced by cs: upon return, no assumption can be made on its value.

9.1.4.113 int ppl_Polyhedron_add_constraints_and_minimize (ppl_Polyhedron_t *ph*, ppl_ConSys_t *cs*)

Adds the system of constraints cs to the system of constraints of ph. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return, ph is guaranteed to be minimized.

Warning:

This function modifies the constraint system referenced by cs: upon return, no assumption can be made on its value.

9.1.4.114 int ppl_Polyhedron_add_generators (ppl_Polyhedron_t *ph*, ppl_GenSys_t *gs*)

Adds the system of generators `gs` to the system of generators of `ph`.

Warning:
This function modifies the generator system referenced by `gs`: upon return, no assumption can be made on its value.

9.1.4.115 int ppl_Polyhedron_add_generators_and_minimize (ppl_Polyhedron_t *ph*, ppl_GenSys_t *gs*)

Adds the system of generators `gs` to the system of generators of `ph`. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return, `ph` is guaranteed to be minimized.

Warning:
This function modifies the generator system referenced by `gs`: upon return, no assumption can be made on its value.

9.1.4.116 int ppl_Polyhedron_add_dimensions_and_embed (ppl_Polyhedron_t *ph*, unsigned int *d*)

Adds `d` new dimensions to the space enclosing the polyhedron `ph` and to `ph` itself.

9.1.4.117 int ppl_Polyhedron_add_dimensions_and_project (ppl_Polyhedron_t *ph*, unsigned int *d*)

Adds `d` new dimensions to the space enclosing the polyhedron `ph`.

9.1.4.118 int ppl_Polyhedron_remove_dimensions (ppl_Polyhedron_t *ph*, unsigned int *ds*[ ], unsigned int *n*)

Removes from `ph` and its containing space the dimensions that are specified in first `n` positions of the array `ds`. The presence of duplicates in `ds` is innocuous.

9.1.4.119 int ppl_Polyhedron_remove_higher_dimensions (ppl_Polyhedron_t *ph*, unsigned int *d*)

Removes the higher dimensions from `ph` and its enclosing space so that, upon successful return, the new space dimension is `d`.

9.1.4.120 int ppl_Polyhedron_add_dimensions_and_constraints (ppl_Polyhedron_t *ph*, ppl_ConSys_t *cs*)

First increases the space dimension of `ph` by adding as many dimensions as is the space dimension of `cs`; then adds to the system of constraints of `ph` a renamed-apart version of the constraints in `cs`.

Warning:
This function modifies the constraint system referenced by `cs`: upon return, no assumption can be made on its value.

9.1.4.121 int ppl_Polyhedron_affine_image (ppl_Polyhedron_t *ph*, unsigned int *var*, ppl_const_LinExpression_t *le*, ppl_const_Coefficient_t *d*)

Transforms the polyhedron `ph`, assigning an affine expression to the specified variable.

Parameters:

ph The polyhedron that is transformed.

var The variable to which the affine expression is assigned.

le The numerator of the affine expression.

d The denominator of the affine expression.

9.1.4.122 int ppl_Polyhedron_affine_preimage (ppl_Polyhedron_t *ph*, unsigned int *var*, ppl_const_LinExpression_t *le*, ppl_const_Coefficient_t *d*)

Transforms the polyhedron `ph`, substituting an affine expression to the specified variable.

Parameters:

ph The polyhedron that is transformed.

var The variable to which the affine expression is substituted.

le The numerator of the affine expression.

d The denominator of the affine expression.

9.1.4.123 int ppl_Polyhedron_shrink_bounding_box (ppl_const_Polyhedron_t *ph*, void(∗ *set_empty*)(void), void(∗ *raise_lower_bound*)(unsigned int k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d), void(∗ *lower_upper_bound*)(unsigned int k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d))

Use `ph` to shrink a generic, interval-based bounding box. The bounding box is abstractly provided by means of the parameters,

Parameters:

ph The polyhedron that is used to shrink the bounding box.

set_empty a pointer to a void function with no arguments that causes the bounding box to become empty, i.e., to represent the empty set.

raise_lower_bound a pointer to a void function with arguments `(unsigned int k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d)` that intersects the interval corresponding to the `k`-th dimension with $[n/d, +\infty)$ if `closed` is non-zero, with $(n/d, +\infty)$ if `closed` is zero. The fraction n/d is in canonical form, that is, n and d have no common factors and d is positive, $0/1$ being the unique representation for zero.

lower_upper_bound a pointer to a void function with argument `(unsigned int k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d)` that intersects the interval corresponding to the `k`-th dimension with $(-\infty, n/d]$ if `closed` is non-zero, with $(-\infty, n/d)$ if `closed` is zero. The fraction n/d is in canonical form.

9.1.4.124 int ppl_Polyhedron_relation_with_Constraint (ppl_const_Polyhedron_t *ph*, ppl_const_Constraint_t *c*)

Checks the relation between the polyhedron `ph` with the constraint `c`. If successful, returns a non-negative integer that is obtained as the bitwise or of the bits (chosen among PPL_POLY_CON_RELATION_IS_DISJOINT PPL_POLY_CON_RELATION_STRICTLY_INTERSECTS, PPL_POLY_CON_RELATION_IS_INCLUDED, and PPL_POLY_CON_RELATION_SATURATES) that describe the relation between `ph` and `c`.

9.1.4.125 int ppl_Polyhedron_relation_with_Generator (ppl_const_Polyhedron_t *ph*, ppl_const_Generator_t *g*)

Checks the relation between the polyhedron `ph` with the generator `g`. If successful, returns a non-negative integer that is obtained as the bitwise or of the bits (only PPL_POLY_GEN_RELATION_SUBSUMES, at present) that describe the relation between `ph` and `g`.

9.1.4.126 int ppl_Polyhedron_check_empty (ppl_const_Polyhedron_t *ph*)

Returns a positive integer if `ph` is empty; returns 0 if `ph` is not empty.

9.1.4.127 int ppl_Polyhedron_check_universe (ppl_const_Polyhedron_t *ph*)

Returns a positive integer if `ph` is a universe polyhedron; returns 0 if it is not.

9.1.4.128 int ppl_Polyhedron_is_bounded (ppl_const_Polyhedron_t *ph*)

Returns a positive integer if `ph` is bounded; returns 0 if `ph` is unbounded.

9.1.4.129 int ppl_Polyhedron_bounds_from_above (ppl_const_Polyhedron_t *ph*, ppl_const_Linear_Expression_t *le*)

Returns a positive integer if `le` is bounded from above in `ph`; returns 0 otherwise.

9.1.4.130 int ppl_Polyhedron_bounds_from_below (ppl_const_Polyhedron_t *ph*, ppl_const_Linear_Expression_t *le*)

Returns a positive integer if `le` is bounded from below in `ph`; returns 0 otherwise.

9.1.4.131 int ppl_Polyhedron_is_topologically_closed (ppl_const_Polyhedron_t *ph*)

Returns a positive integer if `ph` is topologically closed; returns 0 if `ph` is not topologically closed.

9.1.4.132 int ppl_Polyhedron_topological_closure_assign (ppl_Polyhedron_t *ph*)

Assigns to `ph` its topological closure.

9.1.4.133 int ppl_Polyhedron_contains_Polyhedron (ppl_const_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Returns a positive integer if `x` contains or is equal to `y`; returns 0 if it does not.

9.1.4.134 int ppl_Polyhedron_strictly_contains_Polyhedron (ppl_const_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Returns a positive integer if `x` strictly contains `y`; returns 0 if it does not.

9.1.4.135 int ppl_Polyhedron_OK (ppl_const_Polyhedron_t *ph*)

Returns a positive integer if `ph` is well formed, i.e., if it satisfies all its implementation variant; returns 0 and perhaps make some noise if `ph` is broken. Useful for debugging purposes.

9.1.5 Variable Documentation

9.1.5.1 unsigned int PPL_POLY_CON_RELATION_IS_DISJOINT

Individual bit saying that the polyhedron and the set of points satisfying the constraint are disjoint.

9.1.5.2 unsigned int PPL_POLY_CON_RELATION_STRICTLY_INTERSECTS

Individual bit saying that the polyhedron intersects the set of points satisfying the constraint, but it is not included in it.

9.1.5.3 unsigned int PPL_POLY_CON_RELATION_IS_INCLUDED

Individual bit saying that the polyhedron is included in the set of points satisfying the constraint.

9.1.5.4 unsigned int PPL_POLY_CON_RELATION_SATURATES

Individual bit saying that the polyhedron is included in the set of points saturating the constraint.

9.1.5.5 unsigned int PPL_POLY_GEN_RELATION_SUBSUMES

Individual bit saying that adding the generator would not change the polyhedron.

10 PPL Page Documentation

10.1 Prolog Interface

10.1.1 Introduction

The Parma Polyhedra Library comes equipped with a Prolog interface. Despite the lack of standardization of Prolog's foreign language interfaces, the PPL Prolog interface supports several Prolog systems and, to the extent this is possible, provides a uniform view of the library from each such systems.

The system-independent features of the library are described in Section System-Independent Features. Section Compilation and Installation explains how the various incarnations of the Prolog interface are compiled and installed. Section System-Dependent Features illustrates the system-dependent features of the interface for all the supported systems.

10.1.2 System-Independent Features

The Prolog interface provides access to the PPL polyhedra. A general introduction to convex polyhedra, their representation in the PPL and the operations provided by the PPL is given in Sections A Library for Convex Polyhedra, An Introduction to Convex Polyhedra, Representations of Convex Polyhedra, and Operations on Convex Polyhedra of this manual. Here we just describe those aspects that are specific to the Prolog interface.

For proper operation the Prolog interface must be initialized by calling the predicate `ppl_initialize/0` and finalized by calling the predicate `ppl_finalize/0`. Both `ppl_initialize/0` and `ppl_finalize/0` are guarded against multiple invocations so that calling `ppl_initialize/0` several times makes no harm. The same holds for `ppl_finalize/0`. However, the first call to `ppl_initialize/0` must occur before any other predicate of the interface is called. On the other hand, the only interface's predicates callable after `ppl_finalize/0` are `ppl_finalize/0` itself (this further call has no effect) and `ppl_initialize/0`, after which the interface's services are usable again. Some Prolog systems allow the specification of initialization and deinitialization functions in their foreign language interfaces. The corresponding incarnations of the PPL-Prolog interface have been written so that `ppl_initialize/0` and/or `ppl_finalize/0` are called automatically. Section System-Dependent Features will detail in which cases initialization and finalization is automatically performed or is left to the Prolog programmer's responsibility. However, if you want to write portable applications you may decide to invoke `ppl_initialize/0` and `ppl_finalize/0` explicitly: since they can be called multiple times without problems this will result in enhanced portability at a cost that is, by all means, negligible.

The PPL predicates provided by the Prolog interface are specified below. The specification uses the following grammar rules:

```
Topology    --> c | nnc

VarId       --> non-negative integer   variable identifier

PPL_Var     --> '$VAR'(VarId)          PPL variable

LinExpr     --> PPL_Var                PPL variable
           | number                    integer
           | + LinExpr                 unary plus
           | - LinExpr                 unary minus
           | LinExpr + LinExpr         addition
           | LinExpr - LinExpr         subtraction
           | number * LinExpr          multiplication
           | LinExpr * number          multiplication

Constraint  --> LinExpr = LinExpr      equation
           | LinExpr =< LinExpr        nonstrict inequation
           | LinExpr >= LinExpr        nonstrict inequation
           | LinExpr < LinExpr         strict inequation
           | LinExpr > LinExpr         strict inequation

Constraint_System                      list of constraints
            --> []
           | [Constraint]
           | [Constraint | Constraint_System]

Generator   --> point(LinExpr)         point
           | point(LinExpr, number)    point
           | closure-point(LinExpr)    closure point
           | closure-point(LinExpr, number)
                                       closure point
             (Int is the denominator so that the point or
              closure point is defined by Expr/Int.)
           | ray(LinExpr)              ray
           | line(LinExpr)             line

Generator_System                       list of generators
            --> []
           | [Generator]
           | [Generator | Generator_System]

Relation    --> is_disjoint            between a constraint and a polyhedron
           | strictly_intersects       between a constraint and a polyhedron
           | is_included               between a constraint and a polyhedron
           | saturates                 between a constraint and a polyhedron
           | subsumes                  between a generator and a polyhedron
           | nothing

Numerator   --> number | + number | - number

Denominator --> number                 number must be non-zero
```

```
Rational    --> number | + number | - number
           | Numerator/Denominator     rational number

Bound       --> c(Rational)            closed rational limit
           | o(Rational)               open rational limit
           | o(pinf)                   unbounded in the positive direction
           | o(minf)                   unbounded in the negative direction

Interval    --> i(Bound, Bound)        rational interval

Box         --> []
           | [Interval]
           | [Interval | Box]          list of intervals.
```

We first give some general information about using the interface.

- Access to any PPL polyhedron is provided by means of a Prolog term called a *handle*. The data structure of a handle, is implementation-dependent, system-dependent and version-dependent, and, for this reason, deliberately left unspecified. What we do guarantee is that a handle is an ordinary Prolog term that can be used as such and requiring very little memory.

- Only terms bound to *valid* handles may be used to access PPL polyhedra. A handle is made valid by using:

  ```
  ppl_new_Polyhedron_from_dimension/3,
  ppl_new_Polyhedron_empty_from_dimension/3,
  ppl_new_Polyhedron_from_Polyhedron/4,
  ppl_new_Polyhedron_from_constraints/3,
  ppl_new_Polyhedron_from_generators/3.
  ppl_new_Polyhedron_from_bounding_box/3.
  ```

  These predicates will create or copy a PPL polyhedron and construct a valid handle for referencing it. The first argument (in the case of `ppl_new_Polyhedron_from_Polyhedron/4`, the first and third arguments) denotes the topology and can be either `c` or `nnc` indicating a C or NNC polyhedron, respectively. The third argument (in the case of `ppl_new_Polyhedron_from_Polyhedron/4`, the fourth argument) is a Prolog term that is unified with a new valid handle for accessing this polyhedron.

- As soon as a PPL polyhedron is no longer required, the memory occupied by it should be released using the PPL predicate `ppl_delete_Polyhedron/1`. To understand why this is important, consider a Prolog program and a variable that is bound to a Herbrand term. When the variable dies (goes out of scope) or is uninstantiated (on backtracking) the term it is bound to is amenable to garbage collection. But this only applies for the standard domain of the language: Herbrand terms. In Prolog+PPL, when a variable bound to a handle for a PPL Polyhedron dies or is uninstantiated, the handle can be garbage-collected, but the polyhedra to which the handle refers will not be released. Once a handle has been used as an argument in `ppl_delete_Polyhedron/1`, it becomes invalid.

- For a PPL polyhedron with space dimension `k`, the identifiers used for the PPL variables in the constraints and the generators must lie between 0 and $k-1$. Moreover, when using the predicates that combine PPL polyhedra or add constraints or generators to a representation of a PPL polyhedron, the polyhedra referenced and any constraints or generators in the call should follow all the space dimension-compatibility rules stated in Section Representations of Convex Polyhedra.

- As explained above, a polyhedron has a fixed topology C or NNC, that is determined at the time of its initialization. All subsequent operations on the polyhedron must respect all the topological compatibility rules stated in Section Representations of Convex Polyhedra.

- There are a number of predicates whose name ends with `_and_minimize`. These are provided to help the user obtain better performance.

  For some of the operations on polyhedra in the PPL, the internal representation of a polyhedra has to be *minimized*: if it is not already minimized, an extra PPL minimization operation is performed first. However this operation may be very costly and, for this reason, the PPL library is lazy and avoids it as much as it can. For this reason, a predicate without `_and_minimize` ending should be used unless a minimized representation is needed for the next PPL operation. In that case it is more efficient to use the `_and_minimize` predicate. As an example, suppose you have to compute the poly-hull of several polyhedra. Then use the `ppl_Polyhedron_poly_hull_assign/2` for each intermediate step and `ppl_Polyhedron_poly_hull_assign_and_minimize/2` for the last step. If you just have to compute the poly-hull of two polyhedra, then use `ppl_Polyhedron_poly_hull_assign_and_minimize/2`.

`ppl_new_Polyhedron_from_dimension(+Topology, +Integer, -Handle)`

Creates a new universe C or NNC polyhedron $\mathcal{P}$, depending on the value of `Topology`, with `Integer` dimensions. `Handle` is unified with the handle for $\mathcal{P}$. Thus the query

```
?- ppl_new_Polyhedron_from_dimension(c, X, 3).
```

creates the C polyhedron defining the 3-dimensional vector space $\mathbb{R}^3$ with `X` bound to a valid handle for accessing it.

`ppl_new_Polyhedron_empty_from_dimension(+Topology, +Integer, -Handle)`

Creates a new empty C or NNC polyhedron $\mathcal{P}$, depending on the value of `Topology`, with `Integer` dimensions. `Handle` is unified with the handle for $\mathcal{P}$. Thus the query

```
?- ppl_new_Polyhedron_empty_from_dimension(nnc, X, 3).
```

creates an empty NNC polyhedron embedded in $\mathbb{R}^3$ with `X` bound to a valid handle for accessing it.

`ppl_new_Polyhedron_from_Polyhedron(+Topology_1, +Handle_1, +Topology_2, -Handle_2)`

If `Handle_1` refers to a C or NNC polyhedron $\mathcal{P}_1$ (depending on the value of `Topology_1`), then this creates a copy $\mathcal{P}_2$ of $\mathcal{P}_1$ with topology C or NNC, depending on the value of `Topology_2`. `Handle_2` is unified with the handle for $\mathcal{P}_2$. Thus the query

```
?- ppl_new_Polyhedron_empty_from_dimension(nnc, X, 3),
   ppl_new_Polyhedron_from_Polyhedron(c, X, nnc, Y).
```

creates an empty C polyhedron embedded in $\mathbb{R}^3$ referenced by `X` and then makes a copy, converting the topology to an NNC polyhedron. with `Y` bound to a valid handle for accessing it.

When using `ppl_new_Polyhedron_from_Polyhedron/2`, when the source polyhedron is NNC and the copy is C, care must be taken that the source polyhedron referenced by `Handle1` is topologically closed.

ppl_new_Polyhedron_from_constraints(+Topology, +Constraint_System, -Handle)

Creates a polyhedron $\mathcal{P}$ represented by Constraint_System with topology C or NNC, depending on the value of Topology. Handle is unified with the handle for $\mathcal{P}$.

ppl_new_Polyhedron_from_generators(+Topology, +Generator_System, -Handle)

Creates a polyhedron $\mathcal{P}$ represented by Generator_System with topology C or NNC, depending on the value of Topology. Handle is unified with the handle for $\mathcal{P}$.

ppl_new_Polyhedron_from_bounding_box(+Topology, +Box, -Handle)

Creates a polyhedron $\mathcal{P}$ represented by Box with topology C or NNC, depending on the value of Topology, and Handle is unified with the handle for $\mathcal{P}$. A bound of the form o(Rational) can be included in an interval in Box only if Topology is nnc.

ppl_delete_Polyhedron(+Handle)

Deletes the polyhedron referenced by Handle. After execution, Handle is no longer a valid handle for a PPL polyhedron.

ppl_Polyhedron_space_dimension(+Handle, -Integer)

Unifies the space dimension of the polyhedron referenced by Handle with Integer.

ppl_Polyhedron_intersection_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_intersection_assign_and_minimize(+Handle_1, +Handle_2)

Assign to the polyhedron referenced by Handle_1 its intersection with the polyhedra referenced by Handle_2.

ppl_Polyhedron_poly_hull_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_poly_hull_assign_and_minimize(+Handle_1, +Handle_2)

Assign to the polyhedron referenced by Handle_1 its poly-hull with the polyhedra referenced by Handle_2.

ppl_Polyhedron_poly_difference_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_poly_difference_assign_and_minimize(+Handle_1, +Handle_2)

Assign to the polyhedron referenced by Handle_1 its poly-difference with the polyhedron referenced by Handle_2.

`ppl_Polyhedron_H79_widening_assign(+Handle_1, +Handle_2)`

Assigns to the polyhedron referenced by `Handle_1` its H79-widening with the polyhedra referenced by `Handle_2`,

`ppl_Polyhedron_limited_H79_widening_assign(+Handle_1, +Handle_2, +Constraint_System)`

Assigns to the polyhedron referenced by `Handle_1` its H79-widening with the polyhedron referenced by `Handle_2`, limited by the constraints in `Constraint_System`.

`ppl_Polyhedron_topological_closure_assign(+Handle)`

Assigns to the polyhedron referenced by `Handle` its topological closure.

`ppl_Polyhedron_get_constraints(+Handle, -Constraint_System)`

Unifies `Constraint_System` with a list of the constraints in the constraints system representing the polyhedron referenced by `Handle`.

`ppl_Polyhedron_get_minimized_constraints(+Handle, -Constraint_System)`

Unifies `Constraint_System` with a minimized list of the constraints in the constraints system representing the polyhedron referenced by `Handle`.

`ppl_Polyhedron_get_generators(+Handle, -Generator_System)`

Unifies `Generator_System` with a list of the generators in the generators system representing the polyhedron referenced by `Handle`.

`ppl_Polyhedron_get_minimized_generators(+Handle, -Generator_System)`

Unifies `Generator_System` with a minimized list of the generators in the generators system representing the polyhedron referenced by `Handle`.

`ppl_Polyhedron_add_constraint(+Handle, +Constraint)`

Updates the polyhedron referenced by `Handle` to one obtained by adding `Constraint` to its constraint system. Thus, the query

```
?- ppl_new_Polyhedron_from_dimension(c, 3, X),
   A = '$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_add_constraint(X, 4*A + B - 2*C >= 5).
```

will update the polyhedron with handle `X` to consist of the set of points in the vector space $\mathbb{R}^3$ satisfying the constraint $4x + y - 2z >= 5$.

ppl_Polyhedron_add_generator(+Handle, +Generator)

Updates the polyhedron referenced by Handle to one obtained by adding Generator to its generator system. Thus, after the query

```
?- ppl_new_Polyhedron_from_dimension(c, 3, X),
   A = '$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_add_generator(X, point(-100*A - 5*B, 8)).
```

will update the polyhedron with handle X to be the single point $(-12.5, -0.625, 0)^{\mathrm{T}}$ in the vector space $\mathbb{R}^3$.

ppl_Polyhedron_add_constraints(+Handle, +Constraint_System)

Updates the polyhedron referenced by Handle to one obtained by adding to its constraint system the constraints in Constraint_System. E.g.,

```
| ?- ppl_new_Polyhedron_from_dimension(c, 2, X),
     A = '$VAR'(0), B = '$VAR'(1),
     ppl_Polyhedron_add_constraints(X, [4*A + B >= 3, A = 1]),
     ppl_Polyhedron_get_constraints(X, CS).

CS = [4*A+1*B>=3,1*A=1] ?
```

The updated polyhedron referenced by Handle can be empty and a query will succeed even when Constraint_System is unsatisfiable.

ppl_Polyhedron_add_constraints_and_minimize(+Handle, +Constraint_System)

Updates the polyhedron referenced by Handle to one obtained by adding to its constraint system the constraints in Constraint_System. E.g.,

```
?- ppl_new_Polyhedron_from_dimension(c, 2, X),
   A = '$VAR'(0), B = '$VAR'(1),
   ppl_Polyhedron_add_constraints_and_minimize(X, [4*A + B >= 3, A = 1]),
   ppl_Polyhedron_get_constraints(X, CS).

CS = [1*B>= -1,1*A=1]
```

This will fail if, after adding the constraints, the polyhedron is empty. E.g., the following will fail,

```
?- A = '$VAR'(0), B = '$VAR'(1),
   ppl_new_Polyhedron_from_dimension(c, 2, X),
   ppl_Polyhedron_add_constraints_and_minimize(X,
     [4*A + B >= 3, A = 0, B =< 0]),
   ppl_Polyhedron_get_constraints(X, CS).
```

ppl_Polyhedron_add_generators(+Handle, +Generator_System)

Updates the polyhedron referenced by Handle to one obtained by adding to its generator system the generators in Generator_System.

If the system of generators representing a polyhedron is non-empty, then it must include a point (see the paragraph on generator representation in Section Representations of Convex Polyhedra). Thus care must be taken to ensure that, before calling this predicate, either the polyhedron referenced by Handle is non-empty or that whenever Generator_System is non-empty the first element defines a point. E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 3, X),
   A='$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_add_generators(X,
     [point(1*A + 1*B + 1*C, 1), ray(1*A), ray(2*A)]),
   ppl_Polyhedron_get_generators(X, GS).

GS = [ray(2*A), point(1*A+1*B+1*C), ray(1*A)]
```

ppl_Polyhedron_add_generators_and_minimize(+Handle, +Generator_System)

Updates the polyhedron referenced by Handle to one obtained by adding to its generator system the generators in Generator_System.

Unlike the predicate ppl_add_generators, the order of the generators in Generator_System is not important. E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 3, X),
   A='$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_add_generators_and_minimize(X,
     [ray(1*A), ray(2*A), point(1*A + 1*B + 1*C, 1)]),
   ppl_Polyhedron_get_generators(X, GS).

GS = [point(1*A+1*B+1*C), ray(1*A)]
```

ppl_Polyhedron_add_dimensions_and_constraints(+Handle, +Constraint_-System)

After embedding the polyhedron referred to by Handle in a new space that is enlarged by the space dimensions of the constraint system in Constraint_System, it then updates the polyhedron referenced by Handle to one obtained by adding to the new space the constraints in Constraint_System. E.g.,

```
?- ppl_new_Polyhedron_from_dimension(nnc, 2, X),
   A = '$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   D = '$VAR'(3), E = '$VAR'(4),
   ppl_Polyhedron_add_dimensions_and_constraints(X,
        [A > 1, B >= 0, C >= 0]),
   ppl_Polyhedron_get_constraints(P, CS).

CS = [1*C > 1, 1*D >= 0, 1*E >= 0]
```

ppl_Polyhedron_add_dimensions_and_project(+Handle, +Integer)

Projects the polyhedron referred to by Handle onto a space that is enlarged by Integer dimensions, E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 0, X),
   ppl_Polyhedron_add_dimensions_and_project(X, 2),
   ppl_Polyhedron_get_constraints(X, CS),
   ppl_Polyhedron_get_generators(X, GS).

CS = [1*A = 0, 1*B = 0],
GS = [point(0)]
```

ppl_Polyhedron_add_dimensions_and_embed(+Handle, +Integer)

Embeds the polyhedron referred to by Handle in a space that is enlarged by Integer dimensions, E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 0, X),
   ppl_Polyhedron_add_dimensions_and_embed(X, 2),
   ppl_Polyhedron_get_constraints(X, CS),
   ppl_Polyhedron_get_generators(X, GS).

CS = [],
GS = [point(0),line(1*A),line(1*B)]
```

`ppl_Polyhedron_remove_dimensions(+Handle, +List_of_PPL_Vars)`

Removes the dimensions given by the identifiers of the PPL variables in list `List_of_PPL_Vars` from the polyhedron referred to by `Handle`. The identifiers for the remaining PPL variables are renumbered so that they are consecutive and the maximum index is less than the number of dimensions. E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 3, X),
   A='$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_remove_dimensions(X, [B]),
   ppl_Polyhedron_space_dimension(X, K),
   ppl_Polyhedron_get_generators(X, GS).

K = 2,
GS = [point(0),line(1*A),line(1*B),line(0)]
```

`ppl_Polyhedron_remove_higher_dimensions(+Handle, +Integer))`

Projects the the polyhedron referred to by `Handle` onto the first `Integer` dimension. E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 5, X),
   ppl_Polyhedron_remove_higher_dimensions(X, 3),
   ppl_Polyhedron_space_dimension(X, K).

K = 3,
```

The polyhedron $\mathcal{P}$ referenced by `Handle` must have space dimension k greater that or equal to `Integer`.

`ppl_Polyhedron_affine_image(+Handle, +PPL_Var, +LinExpr, +Integer)`

Transforms the polyhedron referenced by `Handle` assigning the affine expression `LinExpr`/`Integer` to `PPL_Var`.

`ppl_Polyhedron_affine_preimage(+Handle, +PPL_Var, +LinExpr, +Integer)`

This is the inverse transformation to that for `ppl_affine_image`.

`ppl_Polyhedron_relation_with_constraint(+Handle, +Constraint, -Relation)`

Unifies `Relation` with the relation the polyhedron referenced by `Handle` has with `Constraint`. The possible relations are listed in the grammar rules above; their meaning is given in Section Operations on Convex Polyhedra. The relation `nothing` means that nothing is known about the relation the polyhedron referenced by `Handle` has with `Constraint`.

`ppl_Polyhedron_relation_with_generator(+Handle, +Generator, -Relation)`

Unifies `Relation` with the relation the polyhedron referenced by `Handle` has with `Generator`. The possible relations are listed in the grammar rules above; The meaning of the relation `subsume` is given in Section Operations on Convex Polyhedra. The relation `nothing` means that nothing is known about the relation the polyhedron referenced by `Handle` has with `Generator`.

`ppl_Polyhedron_check_empty(+Handle)`

Succeeds if and only if the polyhedron referenced by `Handle` is empty.

`ppl_Polyhedron_check_universe(+Handle)`

Succeeds if and only if the polyhedron referenced by `Handle` is the universe.

`ppl_Polyhedron_is_bounded(+Handle)`

Succeeds if and only if the polyhedron referenced by `Handle` is bounded.

`ppl_Polyhedron_bounds_from_above(+Handle, +LinExpr)`

Succeeds if and only if `LinExpr` is bounded from above in the polyhedron referenced by `Handle`.

`ppl_Polyhedron_bounds_from_below(+Handle, +LinExpr)`

Succeeds if and only if `LinExpr` is bounded from below in the polyhedron referenced by `Handle`.

`ppl_Polyhedron_is_topologically_closed(+Handle)`

Succeeds if and only if the polyhedron referenced by `Handle` is topologically closed.

`ppl_Polyhedron_contains_Polyhedron(+Handle_1, +Handle_2)`

Succeeds if and only if the polyhedron referenced by `Handle_1` is included in or equal to the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_strictly_contains_Polyhedron(+Handle_1, +Handle_2)`

Succeeds if and only if the polyhedron referenced by `Handle_1` is included in but not equal to the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_equals_Polyhedron, 2(+Handle_1, +Handle_2)`

Succeeds if and only if the polyhedron referenced by `Handle_1` is equal to the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_get_bounding_box(+Handle, -Box)`

Succeeds if and only if the bounding box of the polyhedron referenced by `Handle` unifies with the box defined by `Box`. E.g.,

```
?- A = '$VAR'(0), B = '$VAR'(1),
   ppl_new_Polyhedron_from_constraints(nnc, [B > 0, 4*A =< 2], P),
   ppl_Polyhedron_get_bounding_box(P, Box).

Box = [i(o(minf), c(1/2)), i(o(0), o(pinf))].
```

Note that the rational numbers in `Box` are in canonical form. E.g., the following will fail:

```
?- A = '$VAR'(0), B = '$VAR'(1),
   ppl_new_Polyhedron_from_constraints(nnc, [B > 0, 4*A =< 2], P),
   ppl_Polyhedron_get_bounding_box(P, Box),
   Box = [i(o(minf), c(2/4)), i(o(0), o(pinf))].
```

10.1.3 Compilation and Installation

When the Parma Polyhedra Library is configured, it tests for the existence of each supported Prolog system. If a supported Prolog system is correctly installed in a standard location, things are arranged so that the corresponding interface is built and installed.

In the sequel, `prefix` is the prefix under which you have installed the library (typically `/usr` or `/usr/local`).

As an option, the Prolog interface can track the creation and disposal of polyhedra. In fact, differently from native Prolog data, PPL polyhedra must be explicitly disposed and forgetting to do so is a very common mistake. To enable this option, configure the library adding `-DPROLOG_TRACK_ALLOCATION` to the options passed to the C++ compiler. Your configure command would then look like

```
path/to/configure --with-cxxflags="-DPROLOG_TRACK_ALLOCATION" ...
```

10.1.4 System-Dependent Features

CIAO Prolog

Support for CIAO Prolog is under development and will be available in a future release.

GNU Prolog

The GNU Prolog interface to the PPL library is available both as “PPL enhanced” GNU Prolog interpreter and as a library that can be linked to GNU Prolog programs. Only GNU Prolog version 1.2.12 or later is supported.

Notice that GNU Prolog version 1.2.12 suffers from a serious limitation as far as foreign code is concerned. In order to be safe you must configure GNU Prolog with the `--disable-ebp` option (note that this has a negative effect on performance). See http://www.cs.unipr.it/pipermail/ppl-devel/2002-June/001777.html, http://www.cs.unipr.it/pipermail/ppl-devel/2002-June/001780.html, http://www.cs.unipr.it/pipermail/ppl-devel/2002-June/001788.html and http://www.cs.unipr.it/pipermail/ppl-devel/2002-June/001789.html for more information.

The ppl_gprolog Executable

If an appropriate version of GNU Prolog is installed on the machine on which you compiled the library, the command `make install` will install the executable `ppl_gprolog` in the directory `prefix/bin`. The `ppl_gprolog` executable is simply the GNU Prolog interpreter with the Parma Polyhedra library linked in. The only thing you should do to use the library is to call `ppl_initialize/0` before any other PPL predicate and to call `ppl_finalize/0` when you are done with the library.

Linking the Library To GNU Prolog Programs

In order to allow linking GNU Prolog programs to the PPL, the following files are installed in the directory `prefix/lib/ppl`: `ppl_gprolog.pl` contains the required foreign declarations; `libppl_gprolog.*` contain the executable code for the GNU Prolog interface in various formats (static library, shared library, libtool library). If your GNU Prolog program is constituted by, say, `source1.pl` and `source2.pl` and you want to create the executable `myprog`, your compilation command may look like

```
gplc -o myprog prefix/lib/ppl/ppl_gprolog.pl source1.pl source2.pl \
  -L '-Lprefix/lib/ppl -lppl_gprolog -Lprefix/lib -lppl -lgmp -lgmpxx -lstdc++'
```

SICStus Prolog

The SICStus Prolog interface to the PPL library is available both as a statically linked module or as a dynamically linked one. Only SICStus Prolog version 3.9.0 or later is supported.

The Statically Linked ppl_sicstus Executable

If an appropriate version of SICStus Prolog is installed on the machine on which you compiled the library, the command `make install` will install the executable `ppl_sicstus` in the directory `prefix/bin`. The `ppl_sicstus` executable is simply the SICStus Prolog system with the Parma Polyhedra library statically linked. The only thing you should do to use the library is to load `prefix/lib/ppl/ppl_sicstus.pl`.

Loading the SICStus Interface Dynamically

In order to dynamically load the library from SICStus Prolog you should simply load `prefix/lib/ppl/ppl_sicstus.pl`. Notice that, for dynamic linking to work, you should have configured the library with the `--enable-shared` option.

SWI-Prolog

The SWI-Prolog interface of the library is available both as a statically linked module or as a dynamically linked one. Only SWI-Prolog version 5.0 or later is supported.

The ppl_pl Executable

If an appropriate version of SWI-Prolog is installed on the machine on which you compiled the library, the command `make install` will install the executable `ppl_pl` in the directory `prefix/bin`. The `ppl_pl` executable is simply the SWI-Prolog shell with the Parma Polyhedra library statically linked: from within `ppl_pl` all the services of the library are available without further action.

Loading the SWI-Prolog Interface Dynamically

In order to dynamically load the library from SWI-Prolog you should simply load `prefix/lib/ppl/ppl_swiprolog.pl`. This will invoke `ppl_initialize/0` automatically but, at least for SWI-Prolog versions up to 5.0.7, it is the programmer's responsibility to call `ppl_finalize/0`. Alternatively, you can load the library directly with

```
:- load_foreign_library('prefix/lib/ppl/libppl_swiprolog').
```

This will call `ppl_initialize/0` automatically. Analogously,

```
:- unload_foreign_library('prefix/lib/ppl/libppl_swiprolog').
```

will, as part of the unload process, invoke `ppl_finalize/0`.

Notice that, for dynamic linking to work, you should have configured the library with the `--enable-shared` option.

YAP

The YAP Prolog interface to the PPL library is available as a dynamically linked module. Only YAP version 4.3.23 or later is supported.

In order to dynamically load the library from YAP you should simply load `prefix/lib/ppl/ppl_yap.pl`. This will invoke `ppl_initialize/0` automatically; it is the programmer's responsibility to call `ppl_finalize/0` when the PPL library is no longer needed. Notice that, for dynamic linking to work, you should have configured the library with the `--enable-shared` option.

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